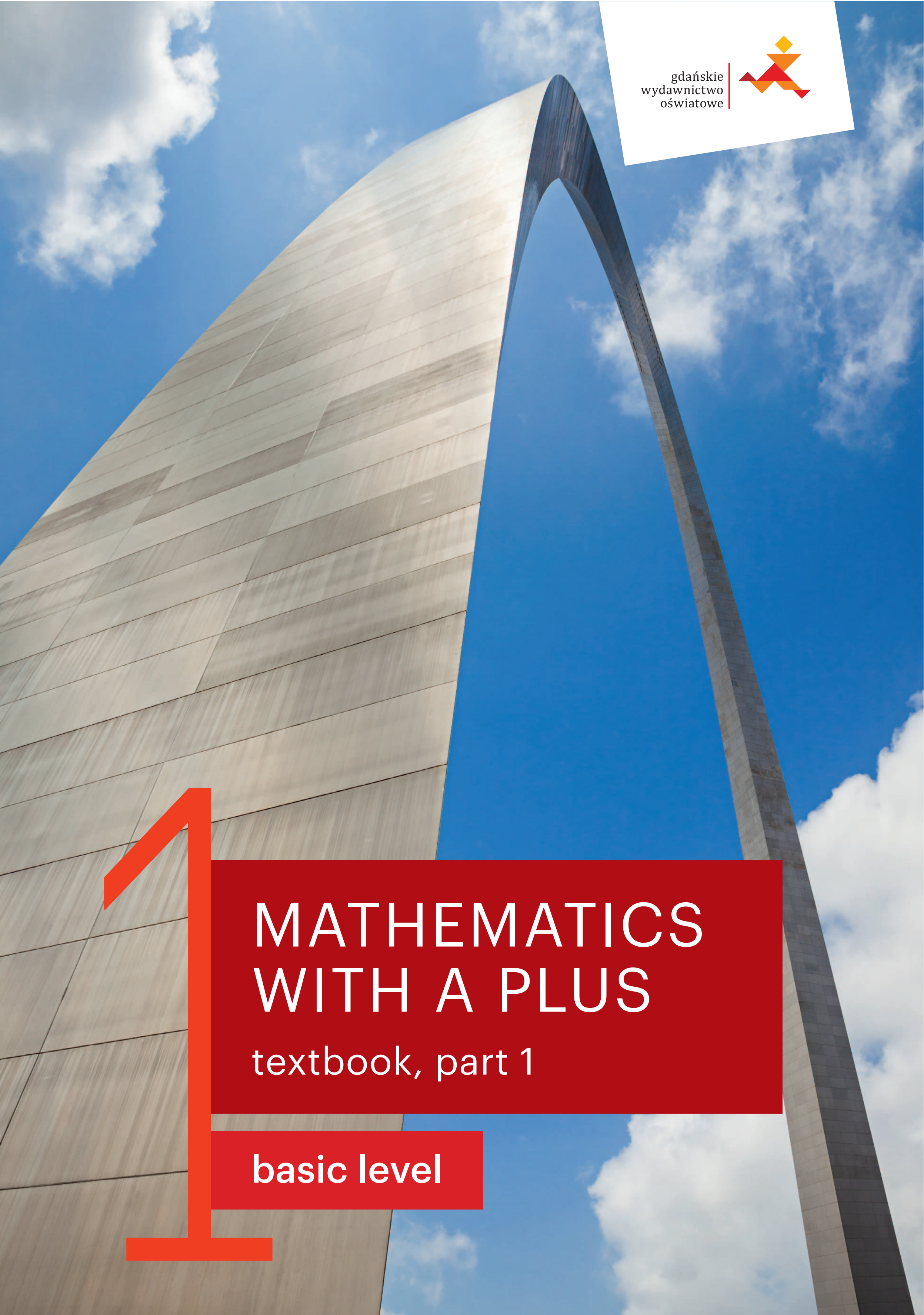

gdańskie wydawnictwo oświatowe

1

MATHEMATICS WITH A PLUS

textbook, part 1

basic level

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Part 1

Sets

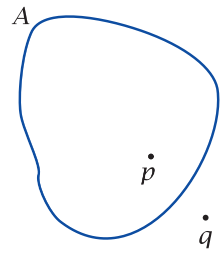
According to the Polish Standard, the hanging mailbox should have a width of 35±0,5 cm, a height of 28,5±0,5 cm and a depth of 20,5 ± 0,5 cm. These conditions can be expressed in the language of mathematics.

Sets and set operations ▪ Intervals

SETS AND SET OPERATIONS

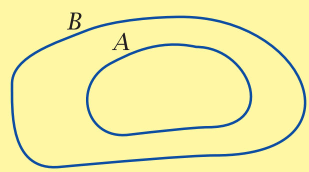
In mathematics, the set is a primitive concept, meaning one that is not defined. We can talk about a set of numbers (e.g. natural numbers, negative numbers), a set of points forming a geometrical figure, a set of figures (e.g. trapezoids, rectangles or obtuse angled triangles), etc.

- ”Objects” from which the set is created, we call elements of this set.
- Sets are usually labeled with upper case letters, and their elements — with lower case letters.
- The sentence: p *belongs to (is an element of) the set* A we can write in the following way: $p \in A$ (the symbol $\in$ reads: *belongs to (is an element of)*).
- The sentence: q *does not belong to (is not an element of) the set* A we can write: $q \notin A$ (the symbol $\notin$ reads: *does not belong to (is not an element of)*).



$\boldsymbol{p \in A}$
$\boldsymbol{q \notin A}$

If all elements of the set A belong to the set B, then we say that set A is contained in set B or that it is a subset of set B. We write it down as: $A \subset B$.



$\boldsymbol{A \subset B}$

A set that has infinitely many elements is called infinite. Other sets are called finite.

EXAMPLES
OF INFINITE SETS

- the set of even natural numbers
- the set of positive numbers
- the set of numbers greater than 5 and also smaller than 7
- the set of prime numbers
- the set of all points of a line
- the set of all rectangles

EXAMPLES
OF FINITE SETS

- the set of two-digit numbers
- the set of negative integers greater than -1000
- the set of divisors of the number 60
- the set of intersection points of one hundred different straight lines
- the set of solutions for the equation $2x + 1 = 7$

The set that has no element is called **an empty set**.
We denote such a set with the symbol $\emptyset$.

Note. We assume that the empty set is a finite set.

For any set A, the empty set is its subset, i.e. $\emptyset \subset A$, and also set A is a subset of A, therefore $A \subset A$.

We can sometimes specify sets by listing out their elements. Here are examples:

$A = \{0, 1, 2, 3\}$	the set of natural numbers less than 4 (finite set)
$B = \{...-6, -4, -2, 0, 2, 4, 6, ...\}$	the set of even integers (infinite set)
$C = \{2, 3, 5, 7, 11, 13, 17, 19, ...\}$	the set of prime numbers (infinite set)
$D = \{1, 70, 2, 35, 5, 14, 7, 10\}$	the set of divisors of number 70 (finite set)

EXERCISE A List elements of the following sets:

A — the set of two-digit numbers divisible by 15

B — the set of prime numbers that are even

The following box contains symbols that we will use to denote certain number sets.

$\mathbb{N}$ — the set of natural numbers
$\mathbb{Z}$ — the set of integers
$\mathbb{Q}$ — the set of rational numbers
$\mathbb{IQ}$ — the set of irrational numbers
$\mathbb{R}$ — the set of real numbers

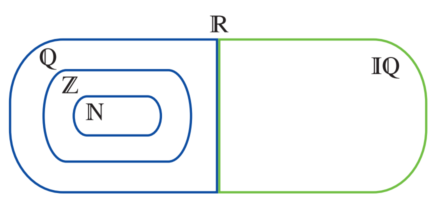
Let us remind you that each rational number can be presented in the form of a ratio of two integers. There are, however, numbers that cannot be represented in this form, e.g. $\sqrt{2}$, $\sqrt{2} - 1$, π. Such numbers are called irrational numbers. All rational and irrational numbers make together the set of real numbers.

We also assume that $\mathbb{R}_+$ is the set of positive real numbers, $\mathbb{R}_-$ — the set of negative real numbers, $\mathbb{Q}_+$ — the set of positive rational numbers, etc.

The relationships between number sets can be described as follows.

$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$

$\mathbb{IQ} \subset \mathbb{R}$



EXERCISE B Select the drawing where the set is shaded of points which:

a) belong to the triangle or the square,

b) belong simultaneously to the triangle and to the square,

c) belong to the triangle, but do not belong to the square.

fig. 1

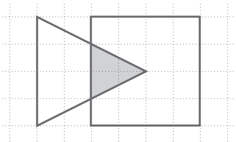


fig. 2

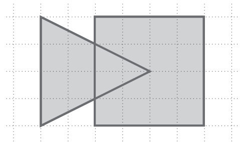
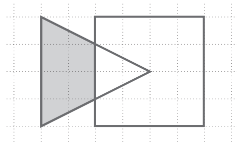


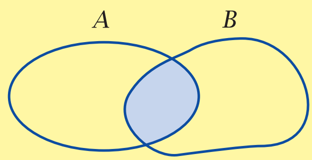
fig. 3



EXERCISE C Draw a triangle and a square, partly overlapping. Shade the set of all points that belong to the square and do not belong to the triangle.

The set, whose all elements belong to set A and also to set B, is called the common part or the **intersection** of sets A and B.

We denote this set $A \cap B$.

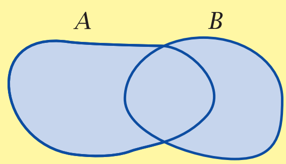


$\boldsymbol{A \cap B}$

If sets A and B do not have common elements, then we say they are **disjoint**. We can express it this way: $A \cap B = \emptyset$.

The set of all elements belonging to set A or to set B (including those that belong to both sets) is called the **union** of sets A and B.

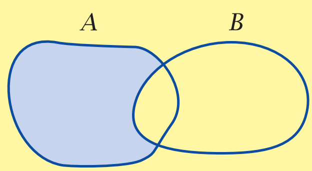
We denote it $A \cup B$.



$\boldsymbol{A \cup B}$

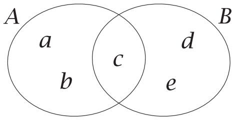
The set composed of all those elements of set A, which do not belong to set B, is called the **difference** of sets A and B.

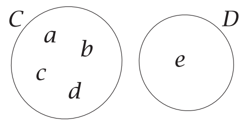
We denote this set $A \setminus B$.

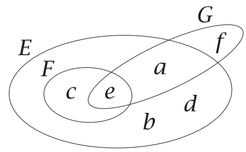


$\boldsymbol{A \setminus B}$

EXAMPLE 1 Elements of the sets, which are shown in the drawings, are letters. Specify the intersection, the union and the difference of the given sets.

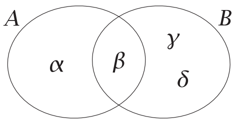






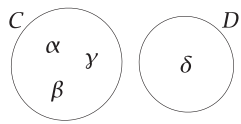
$A \cap B = \{c\}$	$C \cap D = \emptyset$	$E \cap F \cap G = \{e\}$
$A \cup B = \{a, b, c, d, e\}$	$C \cup D = \{a, b, c, d, e\}$	$(E \cup F) \setminus G = \{b, c, d\}$
$A \setminus B = \{a, b\}$	$C \setminus D = C$	$(E \setminus F) \cup G = \{a, b, d, e, f\}$
$B \setminus A = \{d, e\}$	$D \setminus C = D$	$(F \setminus E) \cap G = \emptyset$

PROBLEM Elements of the sets, which are shown in the drawings, are Greek letters. Specify the intersection, the union and the difference of the given sets.



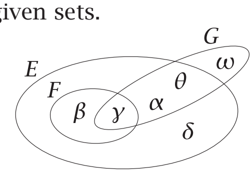
$A \cap B$

$A \setminus B$



$C \cup D$

$D \setminus C$



$(E \cap F) \cup G$

$(G \setminus F) \cap E$

EXAMPLE 2 For the sets:

$$A = \{3, 5, 7, 9\}, \quad B = \{3, 6, 9\}, \quad C = \{4, 6, 7, 9\}$$

specify sets: $A \cap B$, $B \setminus C$, $B \cup C$, $(A \cup B) \setminus C$ and $(B \setminus A) \cup (B \cap C)$.

$A \cap B = \{3, 9\}$ — Elements that belong at the same time to set A and set B are 3 and 9.

$B \setminus C = \{3\}$ — After discarding the elements of set C from set B there remains only 3.

$B \cup C = \{3, 6, 9, 4, 7\}$ — To the elements of set B we add those elements of set C that are not in set B.

$(A \cup B) \setminus C = \{3, 5, 7, 9, 6\} \setminus C = \{3, 5\}$ — We specify the set $A \cup B$, then we reject the elements of set C.

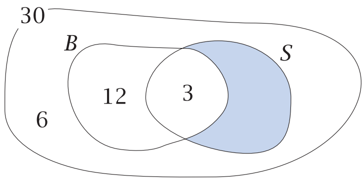
$(B \setminus A) \cup (B \cap C) = \{6\} \cup \{6, 9\} = \{6, 9\}$ — We specify the set $B \setminus A$ and $B \cap C$, then their union.

PROBLEM For the sets specified in the example above, specify:

a) $A \cup C$ c) $C \setminus B$ e) $B \setminus (A \cup C)$

b) $A \cap C$ d) $(A \cap B) \setminus C$ f) $(A \setminus B) \cap C$

EXAMPLE 3 In a 30-person class, there are 6 singles, 12 people have a brother, but no sister, and 3 people have both a brother and a sister. Let B and S mean respectively: the set of people of this class who have a brother, and the set of people who have a sister. How many elements does the set $S \setminus B$ have?



We draw an auxiliary drawing and put the given information on it. (Numbers indicate the number of elements in the relevant sets).

$30 - (6 + 12 + 3) = 9$

We determine the number of elements of the selected set.

Ans. The set $S \setminus B$ has 9 elements.

PROBLEM There are 20 fruits in the basket. Among them, 3 fruits are yellow and acidic, 7 are acidic, but not yellow and 6 — neither yellow nor acidic. Let Y and A denote the set of yellow fruit and the set of acidic fruit, respectively. How many fruits belong to the $Y \setminus A$ set, and how many to $Y \cup A$?

PROBLEMS

1. Determine how many elements the given set has.

a) The set of natural numbers that meet the condition $x \leqslant \sqrt{5}$.

b) The set of integers satisfying the $-6{,}7 \leqslant x \leqslant 1$ condition.

c) The set of negative integers satisfying the condition $-11{,}3 < x \leqslant 3{,}7$.

d) The set of natural numbers that do not meet the condition $x > 20$.

2. a) Which of the given sets are subsets of the set $\{K, R, A, B\}$?

$\{A\}$ $\quad \{B, A, R\}$ $\quad \emptyset$ $\quad \{B, A, S\}$ $\quad \{K, A, R, B\}$ $\quad \{B, A, R, O, K\}$

b) Write several other sets that are included in the set $\{K, R, A, B\}$.

3. Letters are labels of the following sets of figures in the plane:

S — the set of squares	R — the set of rectangles
T — the set of triangles	P — the set of parallelograms
RB — the set of rhombuses	TP — the set of isosceles trapeziums

It is known that each square is a rectangle, which can be written as: $S \subset R$. Write with the character $\subset$ other relationships between the given sets.

4. Determine how many elements the given sets have.

$$A = \{1, 3, 5, 7, 9, 11, \dots, 29, 31\} \qquad B = \{21, 22, 23, 24, \dots, 50\}$$

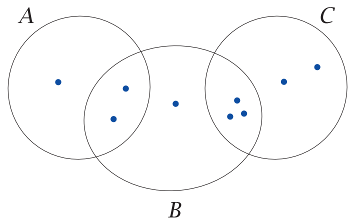
5. Write down:

a) a finite or infinite subset of the set $\mathbb{N}$, in which all elements are even numbers,

b) a four-element subset of the set $\mathbb{IQ}$,

c) an infinite subset of the set $\mathbb{Q}_{-}$.

6. In the figure next, all elements of sets A, B and C are indicated with dots. Count how many elements has the set:



a) $A \cup B$ **d)** $A \cup C$ **g)** $A \cap (B \cup C)$

b) $A \cap C$ **e)** $B \setminus (A \cup C)$ **h)** $C \setminus (A \cap B)$

c) $B \setminus A$ **f)** $(A \cap B) \setminus C$ **i)** $(B \setminus C) \cup A$

7. List all elements of the given set, assuming that:

$A = \{a, b, c\}$ $B = \{b, c, d\}$ $C = \{a, d, e\}$ $D = \{e, f, g\}$

a) $A \cap B$ **c)** $C \cup D$ **e)** $A \cup C$ **g)** $A \cup (B \cap C)$

b) $B \setminus D$ **d)** $A \setminus D$ **f)** $B \cap D$ **h)** $(A \cup C) \setminus D$

A natural number n that has exactly two divisors is called *prime number.*

The first prime numbers are: 2, 3, 5, 7, 11, …

A natural number that has more than two divisors we call a *composite number.*

Numbers 0 and 1 are neither prime not composite.

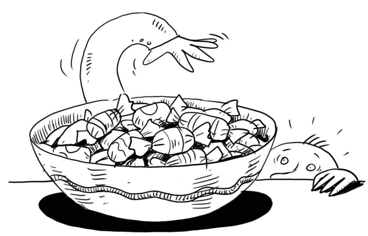
8. Let L_p be the set of prime numbers, and L_c — the set of composite numbers. Determine the set:

a) $\{0, 1, 2, \dots, 19, 20\} \cap L_p$

b) $\left(\{0, 1, 2, \dots, 10\} \setminus L_c\right) \setminus L_p$

c) $\mathbb{N}_{+} \setminus (L_p \cup L_c)$

9. 20 candies were put in a bowl, 11 of them are chocolate and five of chocolate candies contain nuts. Two candies contain nuts, but they are not chocolate. Let's label with C the set of chocolate candies, and with O — the set of candies with nuts. How many elements are in the $C \cup O$ set?



10. In Krasnoland, every citizen is beautiful or rich. The rich constitute 50% of the population. The beautiful Krasnolandians is 50 thousand (of which 70% are poor). How many inhabitants does Krasnoland have? What percentage of Krasnoland residents are beautiful and rich at the same time?

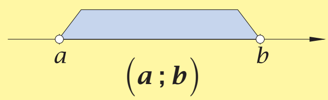
INTERVALS

EXERCISE A Mark on the number line the set of real numbers that meet the inequality $x > 3$, and then the set of numbers that meet the inequality $x < 7$. Mark the set of real numbers that meet the double inequality $3 < x < 7$.

The set of real numbers greater than number a and at the same time smaller than number b is called an **open interval** with the ends a and b.

We denote it with the symbol $(a;b)$.

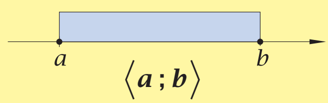
$$\boldsymbol{x \in (a;b), \text{ when } a < x < b}$$



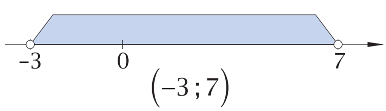
The set of real numbers greater than number a or equal a and at the same time smaller than number b or equal b is called a **closed interval** with the ends a and b.

We denote it with the symbol $\langle a;b \rangle$.

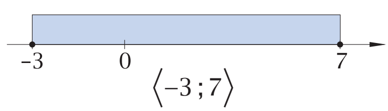
$$\boldsymbol{x \in \langle a;b \rangle, \text{ when } a \leqslant x \leqslant b}$$



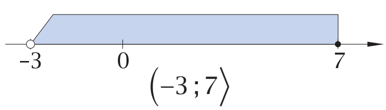
Different types of intervals with the ends –3 and 7 are shown below.



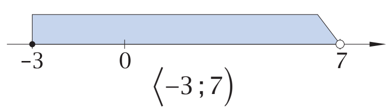
open interval with ends –3 and 7



closed interval with ends –3 and 7



left-side open
and right-side closed interval
with ends –3 and 7



left-side closed
and right-side open interval
with ends –3 and 7

EXERCISE B Draw a number line and mark the intervals on it:

$\left(-3;2\right\rangle$ $\left\langle 3;7\right\rangle$ $\left\langle -10;-8\right)$ $\left(8;9\right)$

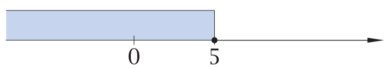
The set of real numbers greater than a is an example of an **unlimited interval**.

We denote it with the symbol $\left(a;+\infty\right)$.

$\boldsymbol{x \in \left(a;+\infty\right)}$, **when** $\boldsymbol{x > a}$

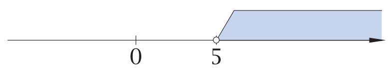
$\left(\boldsymbol{a};+\infty\right)$

Unlimited intervals can be one-sidedly open or closed. For example: $\left\langle a;+\infty\right)$, $\left(-\infty;a\right)$, $\left(-\infty;a\right\rangle$.



$\left(-\infty;5\right\rangle$

right-side closed interval
from minus infinity to 5



$\left(5;+\infty\right)$

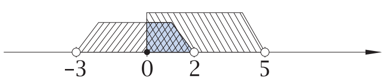
open interval
from 5 to plus infinity

EXERCISE C Mark on the number line the interval $\left\langle 2;+\infty\right)$ and the interval $\left(-\infty;3\right)$.

Intervals are sets, so we can determine their union, difference and intersection.

EXAMPLE Write each of the given sets in a simpler form.

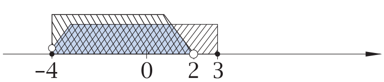
a) $\left(-3;2\right) \cap \left\langle 0;5\right)$



We present both intervals on the number line and highlight their intersection with color.

$\left(-3;2\right) \cap \left\langle 0;5\right) = \underline{\left\langle 0;2\right)}$

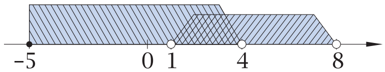
b) $\left\langle -4;2\right) \cap \left(-4;3\right\rangle$



We present both intervals on the number line and highlight their intersection with color.

$\left\langle -4;2\right) \cap \left(-4;3\right\rangle = \underline{\left(-4;2\right)}$

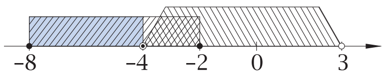
c) $\langle -5\,;4) \cup (1\,;8)$



We present both intervals on the number line and highlight their union with color.

$\langle -5\,;4) \cup (1\,;8) = \underline{\langle -5\,;8)}$

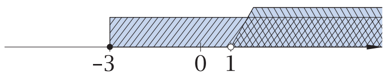
d) $\langle -8\,;-2\rangle \setminus (-4\,;3)$



We present both intervals on the number line and mark with color the set of numbers that belong to the first interval and do not belong to the second one.

$\langle -8\,;-2\rangle \setminus (-4\,;3) = \underline{\langle -8\,;-4\rangle}$

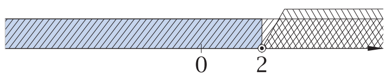
e) $\langle -3\,;+\infty) \cup (1\,;+\infty)$



We present both intervals on the number line and highlight their union with color.

$\langle -3\,;+\infty) \cup (1\,;+\infty) = \underline{\langle -3\,;+\infty)}$

f) $\mathbb{R} \setminus (2\,;+\infty)$



We present on the number line the set $(2\,;+\infty)$ and highlight the set of those numbers that do not belong to it.

$\mathbb{R} \setminus (2\,;+\infty) = \underline{(-\infty\,;2\rangle}$

PROBLEM Mark on the number line and write down in the in the simplest form the set:

a) $\langle -8;7) \cup \langle 2;9\rangle$

b) $(-10;5) \cap \langle -12;4\rangle$

c) $\langle -4;6\rangle \setminus (-3;7)$

d) $(-\infty;6\rangle \cap (-3;+\infty)$

PROBLEMS

1. Write down as an interval the set of all numbers fulfilling the condition:

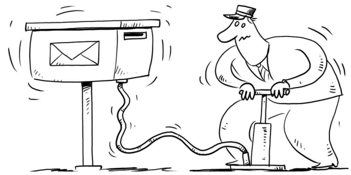
a) $0 \leqslant x < 5$ **b)** $-6 < x < 6$ **c)** $x \geqslant -3$ **d)** $x < 6$

2. Mark the interval on the number line. Define it using the inequality symbols.

a) $\langle -4;5)$ **c)** $(-12;-8\rangle$ **e)** $(-\infty;5)$ **g)** $(-6;+\infty)$

b) $(3;7)$ **d)** $\langle 0;6\rangle$ **f)** $\langle 2;+\infty)$ **h)** $(-\infty;-1\rangle$

3. According to Polish Standard, the hanging mailbox should have a width of $35 \pm 0{,}5$ cm, a height of $28{,}5 \pm 0{,}5$ cm and a depth of $20{,}5 \pm 0{,}5$ cm. In what intervals the dimensions of the mailbox should be?



4. Mark the given set on the number line:

a) $\langle -5;-4) \cap \langle -3;2\rangle$ **c)** $(-3;7\rangle \cup \langle 6;8\rangle$ **e)** $(-\infty;4) \setminus (0;8)$

b) $(-\infty;-2) \cap \langle -3;0\rangle$ **d)** $(-1;5\rangle \cup \langle 2;+\infty)$ **f)** $\langle -2;4) \setminus \langle -1;0)$

5. Write in the easier form.

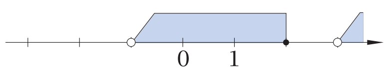
a) $(-4;5\rangle \cap \langle 2;+\infty)$ **d)** $(-\infty;3) \cup \langle 0;6\rangle$ **g)** $(-5;-3) \setminus \langle -4;7)$

b) $\langle 3;10\rangle \cap (-6;1)$ **e)** $(-10;4) \cup (-7;7\rangle$ **h)** $(-\infty;2\rangle \setminus (-\infty;0\rangle$

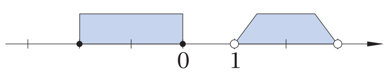
c) $(3;8) \cap \langle 3;6\rangle$ **f)** $(3;+\infty) \cup \langle -2;+\infty)$ **i)** $(2;7) \setminus (2;5)$

6. On the number line, a number set has been marked. Write down as a union of intervals the set of all numbers that do not belong to this set.

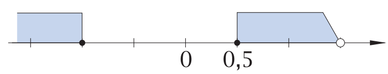
a)



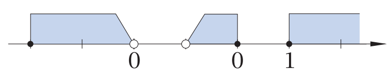
b)



c)



d)



7. Write in another form.

a) $\mathbb{Z} \cap \langle -3;2)$ **d)** $\mathbb{R}_+ \cup \langle -20;3)$ **g)** $\mathbb{R}_- \cup (-7;4\rangle$

b) $\mathbb{N} \setminus (-\infty;5\rangle$ **e)** $\mathbb{Q}_- \setminus (2;+\infty)$ **h)** $\mathbb{Z}_+ \cap (-\infty;3\rangle$

c) $\mathbb{N} \setminus (3;+\infty)$ **f)** $\mathbb{N} \cap (-10;5)$ **i)** $\mathbb{Q}_+ \cup \langle -2;+\infty)$

8. Determine how many integers belong to the given set.

a) $\langle -7{,}5;5)$ **c)** $\mathbb{R} \setminus \left[\left(-\infty;-3\tfrac{1}{3}\right) \cup \left(-1\tfrac{1}{3};+\infty\right) \right]$

b) $(-\infty;200\rangle \cap \langle -200;+\infty)$ **d)** $(-101;10{,}1\rangle \setminus \mathbb{N}$

Algebraic expressions

BMI (body mass index) of a person, who weighs m kilograms and has h meters of height, is calculated from the formula $\mathrm{BMI}=\frac{m}{h^2}$.
It is usually assumed that the body mass is correct when the BMI is greater than 20 and less than 25. Can you determine your BMI?

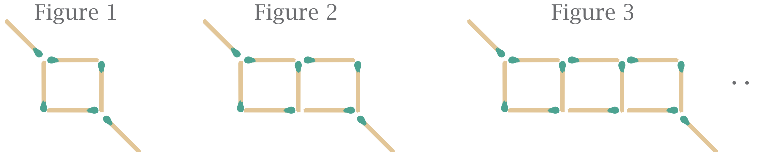
Writing and transforming algebraic expressions ▪ Taking out a common factor ▪ Abridged multiplicaton formulas ▪ Converting formulas ▪ Theorems. Proving

WRITING AND TRANSFORMING ALGEBRAIC EXPRESSIONS

EXERCISE A Simplify to the simplest form the expression $2x+6-a+\frac{8x}{2}-3(x-2a)$ and calculate its value for $x=\frac{1}{6}$ and $a=-\frac{1}{4}$.

EXERCISE B Look at the drawing. The three figures are made of matches according to a certain rule. In the table, what expressions should the question marks be replaced with?

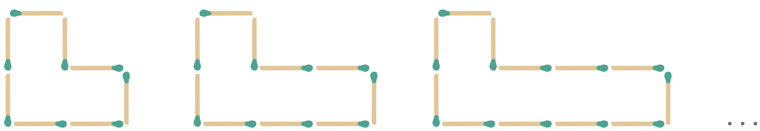
Figure 1 Figure 2 Figure 3



The number of the figure	1	2	3	4	5	n
Number of matches	6	$6+3$	$6+2\times 3$	?	?	?

Look at the next figures made of matches. How many matches of will the fourth, fifth, nth one be arranged?

Figure 1 Figure 2 Figure 3



In the above exercise, it was necessary to formulate general rules according to which matches were arranged.

Such generalizations, recorded using algebraic expressions, are very common in mathematics and other areas of knowledge. E.g:

- The area of an equilateral triangle with the side of length a is: $\frac{a^2\sqrt{3}}{4}$.
- The number of diagonals in a polygon with n sides is $\frac{1}{2}n(n-3)$.
- The dose of Winkristyna medicine for a child that weighs m kilograms ($m>21$), should be $0{,}03m+0{,}6$ milligrams a day.

Examples of algebraic expressions:

$n+2$

$-2x^2y^7$

$\frac{(a+b)h}{2}$

mgh

a^2-b^2

$3(a+b)-2c+7$

$x+5x-\frac{1}{2}x$

$-3x^2y+x^2y$

EXAMPLE Present the given expressions in the form of a monomial or the simplest algebraic sum.

a) $x + 5x - \frac{1}{2}x = 5{,}5x$

We reduce similar terms.

b) $3x(2x + y) - 5(x^2 - 2xy + 3) =$

$= \underline{6x^2} + \underline{\underline{3xy}} - \underline{5x^2} + \underline{\underline{10xy}} - 15 =$

$= x^2 + 13xy - 15$

We multiply monomials by sums in parentheses and reduce similar terms.

c) $(2a + b)(a - 3b + 1) =$

$= 2a^2 - \underline{6ab} + 2a + \underline{ba} - 3b^2 + b =$

$= 2a^2 - 3b^2 - 5ab + 2a + b$

Each component of the first sum is multiplied by each component of the second sum. Then we reduce similar terms.

d) $\frac{2p}{3} - \frac{3 - 5p}{15} = \frac{5 \times 2p}{15} - \frac{3 - 5p}{15} =$

$= \frac{10p - (3 - 5p)}{15} = \frac{10p - 3 + 5p}{15} =$

$= \frac{15p - 3}{15} = p - \frac{1}{5}$

We bring fractions to the common denominator, write the expression in the form of one fraction and reduce similar terms.

PROBLEM Write down as a monomial or an algebraic sum.

a) $3a - a - \frac{1}{6}a$

b) $7ab^2 - 3ab^2 + \frac{ab^2}{2}$

c) $a(2b - 2) - b(a - 2)$

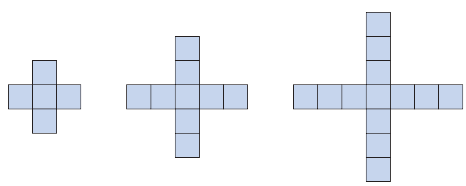
d) $(xy + 2)(x + y - 3)$

e) $\frac{4w - 5}{10} + \frac{3w + 2}{5}$

f) $\frac{1 - 2n}{7} - \frac{5n - 3}{14}$

PROBLEMS

1. Look at the drawings. With how many squares the first figure was built, with how many the second, and with how many the third? Give the observed pattern. With how many squares should be built the fourth, and with how many the n-th figure?



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2. Write down the adequate algebraic expression:

a) half of the sum of numbers a and b,

b) the number 5 times greater than the sum of numbers a and b,

c) the number 4 times smaller than the square of number n,

d) the number 3 less than a half of number x.

3. a) Write the numbers opposite to the numbers:

$3p \qquad -n \qquad b-2c \qquad x\sqrt{2}-7$

b) Write the inverse of numbers:

$\frac{r}{5} \qquad -\frac{1}{y} \qquad x \qquad \frac{3}{7}p \qquad 3a-2b$

The number **opposite** to number a is number $-a$.

For $a \neq 0$, the **inverse** of number a is $\frac{1}{a}$.

An integer is **odd** if it can be expressed in the form

$n = 2k+1$, where $k \in \mathbb{Z}$

4. In what form can a natural number n be written down so that the given condition is fulfilled?

a) The number n is even.

b) The number n is divisible by 5.

c) The remainder from dividing n by 4 is 2.

5. We assume that the numbers a, b, c are positive. Write:

a) 10% of a number a, 130% of a number b, 2% of a number c,

b) the number 40% greater then a, 7% greater than b, 0,5% greater than c,

c) the number 15% smaller than a, 6% smaller than b, 80% smaller than c.

6. a) Write in the form of algebraic expressions the numbers: p% of 27, p% of number a and a number p% greater than a.

b) Prove that p% of number q equals q% of number p.

c) Prove that if the price increases p%, and then q% the result will be the same if it was first increased q% and then p%.

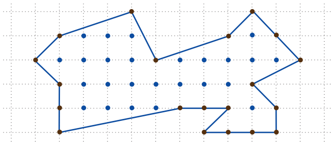
7. A kilogram of apples costs a PLN, pears — p PLN, and oranges — o PLN. Write in the form of algebraic expressions answers to the following questions.

a) Iwona bought 5 kg of apples, 2,5 kg of pears, and oranges, which weighed 78 dag. She paid a fifty-zloty banknote. How much PLN should she receive back?

b) Before closing the store, the price of apples was reduced by 10%, and the price of pears by 20%. The last customer bought 4 kg of apples and 2 kg of pears at reduced prices. How much did he pay? How much more would he pay if he bought the same amount of fruit before lowering prices?

Curiosity

Each vertex of the polygon drawn next lies at the point of intersection of the lines forming the grid, or corner points.



The area of a polygon with vertices at the corner points (grid polygon), with one box of the grid as the unit of the area, can be calculated so: to the number of corner points located inside the polygon we add half of the number of corner points located on the boundary of the polygon and subtract 1. The area of this polygon is thus $25 + \frac{20}{2} - 1 = 34$.

This rule of calculating grid-polygon area was discovered by Austrian mathematician Georg Pick in 1899.

8. Using the Curiosity, write the Pick's algorithm as a formula. Take the designations: i — the number of grid points inside the polygon, b — the number of grid points on the boundary of the polygon.

a) Draw on a checkered paper a triangle with vertices at grid points. Calculate its area in two ways: using the formulas known to you from the geometry and using the Pick's formula.

b) Draw three arbitrary polygons with vertices in grid points and calculate their areas using the Pick's way.

9. Mr. de Fraudant went on a business trip from Paris to Cartouse. He took with him x euros of private money and twice as much business money. He put his private money in his left pocket and his business money in his right pocket.

On Monday, Mr. de Fraudant did not spend anything, but he put 300 € of business papers into the left pocket by mistake. On Tuesday he paid 15 € in the restaurant from his left pocket, and in the afternoon, again by mistake, he transferred the fourth part of the money from his right pocket to the left one. In which pocket he had more money then and how much more?

10. Grandfather brought his grandchildren a sack, in which there were n candies, and said:

— *Let Marek take* $\frac{1}{10}$ *of all the sweets. Ania take* $\frac{1}{10}$ *of what's left, and also 10 times less candies than Marek has.*

— *Now me* — Tosia said. — *I will take* $\frac{1}{10}$ *of what has been left, and in addition, 10 times less sweets than Marek and Ania have together.*

The other candies were distributed by the children themselves according to the same rule. When the last child took his/her portion, the candies were over. How many children were there?

TAKING OUT A COMMON FACTOR

Some products of algebraic expressions can be transformed to obtain an algebraic sum. When all the terms of an algebraic sum have a common factor, we can perform the reverse operation.

EXERCISE A What expression should the asterisk be replaced with?

a) $\star \times (x+4) = 3x+12$

b) $\star \times (a-5) = a^2 - 5a$

c) $\star \times (x+y) = 4x^2 + 4xy$

d) $\star \times (a^2 + 2ab) = a^3 + 2a^2b$

EXAMPLE 1 Take out a common factor.

a) $15x^2 - 20xy = \underbrace{5x} \times 3x - \underbrace{5x} \times 4y =$

$= 5x(3x - 4y)$

A common factor of the monomials $15x^2$ and $-20xy$ is the monomial $5x$.

b) $8m^2n + 6m^3 + 2m =$

$= \underbrace{2m} \times 4mn + \underbrace{2m} \times 3m^2 + \underbrace{2m} \times 1 =$

$= 2m(4mn + 3m^2 + 1)$

A common factor in $8m^2n$, $6m^3$ and $2m$ is the monomial $2m$.

PROBLEM Take out a common factor.

a) $12p + 20p^2$ b) $3ab - 15a$ c) $4xy + 6x - 10x^2$

Sometimes the common factor can be in the form of an algebraic sum.

EXERCISE B Indicate the common factor and present the expression in the form of a product.

a) $(x+7)(y+7) + (x+7)(y-4)$ b) $(5m-2)(3n+1) - (5m-2)(2n-1)$

EXAMPLE 2 Present in the form of a product.

$a^2 - 2a + ab - 2b = a\underbrace{(a-2)} + b\underbrace{(a-2)}$

$= (a-2)(a+b)$

In each of the two groups of terms: $a^2 - 2a$ and $ab - 2b$ we take out a common factor.

The common factor in the expressions $a(a-2)$ and $b(a-2)$ is $a-2$.

PROBLEM Present in the form of a product:

a) $ab - 2a + 5b - 10$ b) $3x^2 + x - 3xy - y$

Note. When looking for a common factor, terms can usually be grouped in different ways. For example, the expression from Example 2 can be transformed like this: $a^2 - 2a + ab - 2b = a^2 + ab - 2a - 2b = a(a+b) - 2(a+b) = (a+b)(a-2)$

EXAMPLE 3 Present in the form of a product.

$2ab - 15 - 3b + 10a =$

$= 2ab + 10a - 3b - 15 =$

$= 2a(b+5) - 3(b+5) = (b+5)(2a-3)$

We find groups of terms so that in each of them we can take out a common factor.

The common factor of expressions $2a(b+5)$ and $-3(b+5)$ is $b+5$.

PROBLEM Present in the form of a product.

a) $2mn + 6m + 3n + 9$ b) $3p^2 - 3q + 9p - pq$

PROBLEMS

1. Present in the form of a product.

a) $3x + 15y$ **c)** $17x^2 + 17$ **e)** $xy + 3x$ **g)** $9xy - y$

b) $7x + 14y$ **d)** $bc - bd$ **f)** $py - p$ **h)** $ab + 3b^2$

2. Calculate cleverly (take out a common factor).

a) $28 \times 17 - 28 \times 15$ **c)** $31 \times 15 - 62 \times 7$ **e)** $13 \times 49 - 26 \times 27$

b) $134 \times 135 - 133 \times 135$ **d)** $12 \times 13 - 6 \times 26$ **f)** $57^2 - 57 \times 55$

$$\frac{2x+6}{2} = \frac{\cancel{2}(x+3)}{\cancel{2}} = x + 3$$

3. Take out a common factor in the numerator and then simplify the expression.

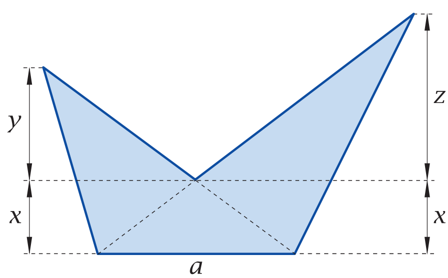
a) $\frac{3x+12}{3}$ **c)** $\frac{-10a+15}{5} + \frac{12a-18}{6}$

b) $\frac{8y-4}{4}$ **d)** $\frac{14x-21}{7} - \frac{64-16x}{8}$

4. Calculate cleverly the value of the expression for the given value of variable x.

a) $x^2 + 0{,}24x$ for $x = 0{,}76$ **b)** $3x^2 - 3x$ for $x = 1{,}1$

5. Show that the area of the pentagon next to equals $\frac{1}{2}a(x+y+z)$.



6. Present in the form of a product.

a) $(x + y) \times a + (x + y) \times b$

b) $3(z - y) - a(z - y)$

c) $(m - 1)(k - 1) + (k - 1)(m + 1)$

d) $(a - 1)(b + c) - (a - 1)(b - c)$

7. Present the algebraic sum in the form of a product.

a) $15mn + 8 - 6n - 20m$

b) $10a^2 + 9b - 6a - 15ab$

c) $3p^2 - 10q + 2p - 15pq$

d) $6x^2 - 2y - 4xy + 3x$

8. Present the algebraic sum in the form of a product.

a) $st^2 - 4s^2t - st + 4s^2$

b) $-9u^2v - 3u + 3uv^2 + v$

c) $8p + 20p^2q - 6q - 15pq^2$

d) $-18y + 12y^2 + 15x - 10xy$

ABRIDGED MULTIPLICATION FORMULAS

EXERCISE A Present in the form of a sum.

a) $(x + 2)(x + 2)$

b) $(5 + x)^2$

c) $(x - 3)^2$

By converting some algebraic expressions, we can use the formulas next to. These are identities called abridged multiplication formulas.

Square of sum:

$$(a + b)^2 = a^2 + 2ab + b^2$$

Square of difference:

$$(a - b)^2 = a^2 - 2ab + b^2$$

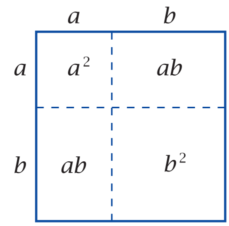
Each of these formulas can be proven by transforming the left side of equality to get the right side. Here is the proof of the formula for the square of sum.

$$\mathrm{L} = (a + b)^2 = (a + b)(a + b) = a^2 + ab + ba + b^2 = a^2 + 2ab + b^2 = \mathrm{R}$$

EXERCISE B Prove the formula for the square of difference.

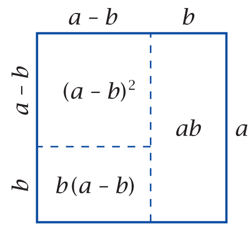
Worth knowing!

The formulas for the sum square and the square of difference for positive numbers can be interpreted geometrically.



In the figure, a square with side length $a + b$ is divided into squares with areas a^2 and b^2 and two rectangles, each with area ab. The area of the whole square is the sum of the areas of all these quadrilaterals. From here we get:

$$(a+b)^2 = a^2 + 2ab + b^2$$



In the next drawing, a square with the side of length a has been divided into a square with an area $(a–b)^2$ and rectangles with areas ab and $b(a–b)$. The area of the smaller square can be calculated by subtracting from the area of the large square the sum of the areas of the rectangles:

$$(a-b)^2 = a^2 - (ab + b\,(a-b))$$

After appropriate transformations we will get:

$$(a-b)^2 = a^2 - 2ab + b^2$$

EXAMPLE 1 Transform the expression.

a) $(4 + 3x)^2 = 4^2 + 2 \times 4 \times 3x + (3x)^2 = 16 + 24x + 9x^2$

b) $\left(\frac{2}{3}x - y^2\right)^2 = \left(\frac{2}{3}x\right)^2 - 2 \times \frac{2}{3}x \times y^2 + (y^2)^2 = \frac{4}{9}x^2 - \frac{4}{3}xy^2 + y^4$

c) $(5 + x)^2 - (1 - 5x)^2 = 25 + 2 \times 5x + x^2 - \left(1 - 2 \times 5x + (5x)^2\right) =$
$= 25 + 10x + x^2 - 1 + 10x - 25x^2 = -24x^2 + 20x + 24$

PROBLEM Using the abridged multiplication formulas transform the expression.

a) $(5 + 6p)^2$ b) $(2x - 3y)^2$ c) $(a - 4b)^2 - (2a + b)^2$

Using the known formulas, we can easily prove the formulas for the cube of sum and cube of difference.

Cube of sum:

$$\boldsymbol{(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3}$$

Cube of difference:

$$\boldsymbol{(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3}$$

Here is the proof of the formula for the cube of sum.

$\mathrm{L} = (a+b)^3 = (a+b)(a+b)^2 =$
$= (a+b)(a^2 + 2ab + b^2) =$
$= a^3 + 2a^2b + ab^2 + a^2b + 2ab^2 + b^3 = a^3 + 3a^2b + 3ab^2 + b^3 = \mathrm{R}$

EXERCISE C Prove the formula for the cube of difference.

EXAMPLE 2 Present the expression in the form of an algebraic sum.

a) $\left(\frac{1}{3}+2p\right)^3 = \left(\frac{1}{3}\right)^3 + 3\times\left(\frac{1}{3}\right)^2\times 2p + 3\times\frac{1}{3}\times(2p)^2 + (2p)^3 = \frac{1}{27} + \frac{2}{3}p + 4p^2 + 8p^3$

b) $\left(2a-\frac{1}{2}b\right)^3 = (2a)^3 - 3(2a)^2\times\frac{b}{2} + 3\times 2a\times\left(\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^3 = 8a^3 - 6a^2b + \frac{3ab^2}{2} - \frac{b^3}{8}$

PROBLEM Present in the form of an algebraic sum.

a) $(3a+5)^3$ b) $(x-2y)^3$

Another formula allows the difference of squares of two numbers represent as a product.

Difference of squares:

$$a^2 - b^2 = (a-b)(a+b)$$

EXERCISE D Prove the formula above.

Worth knowing!

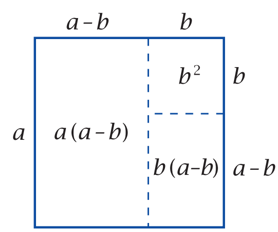
The formula for the difference of squares for positive numbers can also be interpreted geometrically.

When the square with side of length b is subtracted from the square with side of length a, the remaining part will have an area equal to the sum of areas $a(a-b)$ and $b(a-b)$.

$$a^2 - b^2 = a(a-b) + b(a-b)$$

From here we get the formula:

$$a^2 - b^2 = (a-b)(a+b)$$



EXAMPLE 3 a) Present the given algebraic sum in the form of a product.

$9x^2 - 1 = (3x)^2 - 1^2 = (3x-1)(3x+1)$

b) Present the given product in the form of an algebraic sum.

$\left(3+\frac{b}{2}\right)\left(\frac{b}{2}-3\right) = \left(\frac{b}{2}+3\right)\left(\frac{b}{2}-3\right) = \left(\frac{b}{2}\right)^2 - 3^2 = \frac{b^2}{4} - 9$

PROBLEM a) Present the expression $25x^2 - \frac{1}{4}y^2$ in the form of a product.

b) Present the product $\left(2a-\frac{b}{3}\right)\left(2a+\frac{b}{3}\right)$ in the form of an algebraic sum.

The difference of cubes of two numbers can be broken down into factors, one being the difference of those numbers.

Difference of cubes:

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

EXERCISE E Prove the formula given above.

EXAMPLE 4 Present the given algebraic sum in the form of a product.

$$27a^3 - \frac{b^3}{125} = (3a)^3 - \left(\frac{b}{5}\right)^3 =$$

We present the expression as a difference of cubes.

$$= \left(3a - \frac{b}{5}\right)\left((3a)^2 + 3a \times \frac{b}{5} + \left(\frac{b}{5}\right)^2\right) =$$

We break down the expression into factors using the formula above.

$$= \left(3a - \frac{b}{5}\right)\left(9a^2 + \frac{3}{5}ab + \frac{b^2}{25}\right)$$

The result is presented in the simplest form.

PROBLEM Present the expression $1000p^3 - \frac{1}{27}q^3$ in the form of a product.

You can create similar formulas for the differences of higher powers:

$$a^4 - b^4 = (a - b)(a^3 + a^2b + ab^2 + b^3)$$

$$a^5 - b^5 = (a - b)(a^4 + a^3b + a^2b^2 + ab^3 + b^4)$$

$$a^6 - b^6 = (a - b)(a^5 + a^4b + a^3b^2 + a^2b^3 + ab^4 + b^5)$$

EXERCISE F Prove one of those identities.

All the above identities can be written in the form of a generalized formula:

$$a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \ldots + a^2b^{n-3} + ab^{n-2} + b^{n-1})$$

EXAMPLE 5 Present the given expression in the form of a product.

$$32x^5 - 1 = (2x)^5 - 1^5 =$$

We present the expression as the difference of fifth powers.

$$= (2x - 1)\left((2x)^4 + (2x)^3 \times 1 + (2x)^2 \times 1^2 + 2x \times 1^3 + 1^4\right) =$$

We use the formula: $a^5 - b^5 = (a - b)(a^4 + a^3b + a^2b^2 + ab^3 + b^4)$

$$= (2x - 1)(16x^4 + 8x^3 + 4x^2 + 2x + 1)$$

PROBLEM Present the expression $\frac{a^5}{100\,000} - 1$ in the form of a product.

PROBLEMS

1. Write in the form of an algebraic sum.

a) $(5+p)^2$ **b)** $(1-4x)^2$ **c)** $\left(\frac{2}{a}+3a\right)^2$ **d)** $(m^2-2k)^2$ **e)** $\left(\frac{2}{3}y-3\right)^2$

2. Simplify the expression.

a) $(3x+4)^2-x(2+9x)$ **c)** $(5a-b)^2+b(10a-b)$ **e)** $(3m-2)^2+(6m+1)^2$

b) $2p(3-8p)+(1+4p)^2$ **d)** $y(4x+y)-(x+2y)^2$ **f)** $(5a+b)^2-(b-5a)^2$

3. Replace the symbols ♠ and ♣ with such numbers to get an identity that every real number fulfills.

a) $x^2+14x+49=(x+♠)^2$ **d)** $x^2-6x+♣=(x-♠)^2$

b) $a^2-5a+\frac{25}{4}=(a-♠)^2$ **e)** $y^2-4y+♣=(y-♠)^2$

c) $2t^2+20t+50=2(t+♠)^2$ **f)** $9x^2+12x+♣=(3x+♠)^2$

4. Write in the form of an algebraic sum.

a) $(a+b)(b+a)$ **b)** $(-a-b)^2$ **c)** $(-a+b)^2$ **d)** $(-a-b)(a+b)$

5. Write in the form of an algebraic sum the expression $(a-1)^4$.

6. Write in the form of an algebraic sum.

a) $(2+b)^3$ **c)** $\left(\frac{x}{2}+4\right)^3$ **e)** $\left(\frac{1}{w}+3w\right)^3$ **g)** $\left(ac+\frac{c}{3}\right)^3$

b) $(p-0{,}1)^3$ **d)** $\left(5v-\frac{1}{5}\right)^3$ **f)** $\left(\frac{2}{a^2}-\frac{a}{2}\right)^3$ **h)** $\left(\frac{1}{3y}-z^2\right)^3$

7. Write in the form of a product.

a) $49-16x^2$ **c)** $-x^2+\frac{4}{9}y^2$ **e)** $\frac{1}{9}-\frac{a^4}{4}$ **g)** $100m^2-\frac{4}{n^2}$

b) $36a^2-1$ **d)** $0{,}81a^2-\frac{1}{b^2}$ **f)** $25x^2-y^4$ **h)** $\frac{p^2}{121}-0{,}04$

8. Calculate cleverly, using the abridged multiplication formulas.

a) 1005×995 **b)** 510×490 **c)** 207×193

9. Write down in the form of a product.

a) p^3-27 **c)** $1000-k^3$ **e)** $64m^3-\frac{n^3}{8}$ **g)** $a^3-\frac{1}{8}$

b) $\frac{1}{125}-v^3$ **d)** $8a^3-\frac{1}{27}$ **f)** $\frac{w^3}{1000}-\frac{125}{w^3}$ **h)** $0{,}001x^3-\frac{27}{y^3}$

10. Write down in the form of a product.

a) w^5-1 **b)** $1-b^6$ **c)** $16-p^4$ **d)** $\frac{a^5}{32}-1$ **e)** $x^7-\frac{1}{y^7}$

CONVERTING FORMULAS

In some countries, the temperature is measured in Fahrenheit degrees (°F). The relationship between the Celsius and Fahrenheit scales expresses the formula:

$$f = \frac{9}{5}c + 32$$

Using this formula, we can change the temperature c (expressed in Celsius degrees) to the temperature f (in Fahrenheit degrees).

EXERCISE Express in Fahrenheit degrees: 0°C, 100°C, 36,6°C.
Express in the Celsius degrees: −40°F, 95°F, 0°F.

If we want to perform the reverse operation, that is, change the temperature on the Fahrenheit scale for temperature expressed in Celsius, the most comfortable it is first to convert the formula $f = \frac{9}{5}c + 32$, solving it for c.

$$f = \frac{9}{5}c + 32 \quad | - 32$$

$$f - 32 = \frac{9}{5}c \quad | \times \frac{5}{9}$$

$$\frac{5}{9}(f - 32) = c$$

$$c = \frac{5}{9}(f - 32)$$

We swap the sides of the equality.

By converting formulas, we do the same as in solving equations. We can add an expression to, or subtract from, both sides of the equality sign; we can also multiply or divide both sides by the same expression (if its value is different from 0). If the variable the formula is solved for occurs in several positions, the operation of taking out a common factor can be of use.

EXAMPLE Solve the formula $a = \frac{b}{b+1} + 2$ for b (we assume that $b + 1 \neq 0$, i.e. $b \neq -1$).

$$a = \frac{b}{b+1} + 2 \quad | \times (b+1)$$

$$a(b+1) = b + 2(b+1)$$

$$ab + a = b + 2b + 2 \quad | - 3b - a$$

$$ab - 3b = 2 - a$$

We take out the common factor b.

$$b(a-3) = 2 - a \quad | \div (a-3)$$

Assumption: $a \neq 3$.

$$b = \frac{2-a}{a-3}$$

PROBLEM Solve for q the formula: a) $p = \frac{2q}{1-q}$ b) $p = \frac{5}{3+q} + 1$

PROBLEMS

When solving the problems, write down appropriate assumptions for the variables that occur in the expressions.

1. Solve the given equality for a.

a) $b = 5a - \sqrt{2}$

b) $d = 2n - 3a$

c) $k = 2b - a\sqrt{3}$

d) $p = \frac{2a}{b}$

e) $w = \frac{ab}{3c}$

f) $r = \frac{5d}{a}$

2. Solve for x and for y the given equalities.

a) $2y + x = 6$

b) $3x - y = 9$

c) $2x - 5y = 3$

d) $10x + 2y = 5$

e) $3y = 10x - 5$

f) $12x = 4y - 1$

3. Solve the formula for the indicated variable.

a) $m = a(3 + n)$; $\quad n$

b) $u = \frac{2a}{b-1}$; $\quad a$

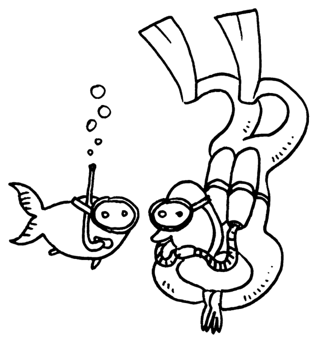
c) $v = \frac{3a - b}{u}$; $\quad b$

d) $w = \frac{2 - 3r}{p}$; $\quad r$

e) $d = \frac{a + b}{2} - c$; $\quad b$

f) $z = \frac{5}{6w - s}$; $\quad s$

4. The pressure under water depends on the depth. The greater depth under the surface of the sea, the higher is the pressure. The relationship between these variables may be described by the formula $p = \frac{1}{10}x + 1$, where p is the pressure expressed in atmospheres (atm), and x — depth in meters.



a) Solve the formula for x.

b) Calculate the pressure at 15 m.

c) At what depth can the diver go down so that the pressure does not exceed 3 atm?

5. Solve the given formula for the indicated variable.

a) $u = 2k - kr$; $\quad k$

b) $3a = 2ab - 1$; $\quad a$

c) $p - 2q = pq + 5$; $\quad q$

d) $p = \frac{r}{r-1}$; $\quad r$

e) $u = \frac{pr}{p+r}$; $\quad p$

f) $\frac{1}{m} = \frac{k}{p+k}$; $\quad k$

g) $f = \frac{3t}{a - 2t} - 1$; $\quad t$

h) $p = 5 - \frac{m}{n+1}$; $\quad n$

i) $x = \frac{2xy - 3}{2y + 2} - 2$; $\quad y$

6. Transform the equality in order to calculate the indicated variable.

a) $(m + n)(m - n) = m(4 + m)$; $\quad m$

b) $\frac{r - p}{r} = \frac{2p + r}{r - p}$; $\quad r$

c) $(u + v)^2 - 4 = u^2 + v^2$; $\quad u$

d) $(k + l)^3 = k(k^2 + 3kl)$; $\quad k$

7. Transform the equality in order to calculate the quotient $\frac{a}{b}$.

a) $a = 4b$

b) $5a - 3b = 0$

c) $\frac{a+3b}{b} = 7$

d) $\frac{5b}{a+b} = 2$

Curiosity

Do you know, how many centimeters longer is the shoe which is one size larger than yours?

In Poland, two numbering systems of shoes are alternatively used: English or French. Shoe numbers 3, $5\frac{1}{2}$, 9, etc. are in English numbering. Sizes 36, $37\frac{1}{2}$, 40, etc. are in the French numbering.

The English system was created at the beginning of the 14th century and has a rather complicated numbering law.

The unit is here $\frac{1}{3}$ inch, or about 8,5 mm.

The French system was created at the end of the 18th century. Law of numbering is much simpler: the unit is $\frac{2}{3}$ cm $\approx$ 6,7 mm.

The relationship between the length of the foot in centimeters and the size of the shoe is described by the following formula:

$$L = \frac{A+25}{3} \times 2{,}54 \qquad L = \frac{2}{3}F$$

L — foot length in centimeters
A — size of the shoe in the English numbering
F — size of the shoe in the French numbering

When applying these formulas in practice, it must be remembered that shoes are produced not in all possible sizes, but only in those that are expressed in a natural number or in a natural number increased by $\frac{1}{2}$.

8. a) Read the Curiosity. What is the length of a foot, which number 37 (in the French scale) is assigned for, and what that, which number 7 (in the English scale) fits?

b) Convert the formulas given in the curiosity so that you can easily calculate the correct size of the shoe when the length of the foot is known (in cm). Measure the length of your foot and calculate which size of shoes in English numbering, and which — in the French numbering is right for you (remember that the result must be with accuracy of 0,5).

c) Find a formula that allows you to calculate the number of the shoe in English numbering, when you know the number of the shoe in the French numbering.

9. Solve for u.

$$r = \cfrac{1}{1 + \cfrac{1}{1+\frac{1}{u}}}$$

THEOREMS. PROVING

Mathematical theorems are often formulated in the form: *If…, then…*. Such a sentence is called an implication; we can write it short using the symbol $\Rightarrow$.

Note. In English, *then* is often omitted.

Implication: *if p, then q* can be written shorter:

$$p \Rightarrow q$$

The first part of the theorem in the form of an implication is called the **assumption** and the second part is the **conclusion**. Here are examples:

$$\text{If } \underbrace{\textit{the side of a square has the length of } a,}_{\text{assumption}} \text{ then } \underbrace{\textit{its diagonal has the length of } a\sqrt{2}.}_{\text{conclusion}}$$

$$\underbrace{a \geqslant 0 \text{ and } b \geqslant 0}_{\text{assumption}} \Rightarrow \underbrace{\sqrt{ab} = \sqrt{a} \times \sqrt{b}}_{\text{conclusion}}$$

$$\underbrace{A \subset B \text{ and } B \subset C}_{\text{assumption}} \Rightarrow \underbrace{A \subset C}_{\text{conclusion}}$$

Even if the theorem is not in the form of an implication, it is usually possible to ”convert” it to implication.

For example, the statement:

The sum of two even numbers is an even number.

you can formulate like this:

$$\text{If } \underbrace{\textit{two numbers are even,}}_{\text{assumption}} \text{ then } \underbrace{\textit{their sum is also an even number.}}_{\text{conclusion}}$$

When proving mathematical theorems, two types of argument for truth are most often used. One of them is called direct proof and the other one is indirect proof.

When a theorem in the form of implication is proved by the direct method, we assume that the assumption is true and we show that the conclusion is also true.

direct proof:

assumption

$\downarrow$

conclusion

EXAMPLE 1 Prove that for any integers a and b the following is true:

If a is a number divisible by 6 and b is a number divisible by 15, then the sum of the numbers a and b is divisible by 3.

Assumption: Number a is divisible by 6 and number b is divisible by 15.
Conclusion: The number $a + b$ is divisible by 3.

Proof

We assume that a is divisible by 6 and b is divisible by 15.

We assume that the assumption is true.

Thus $a = 6m$ and $b = 15n$ for some integers m and n.

In that case:

$a + b = 6m + 15n = 3 \times 2m + 3 \times 5n = 3(2m + 5n)$

The numbers m and n are integers, so $2m+5n$ is also an integer. Number $a+b$ is the product of number 3 and an integer, so it is divisible by 3. □

We have proved that the conclusion is true.

PROBLEM Prove that when we subtract a number divisible by 20 from a number divisible by 8, we get a number divisible by 4.

Note. At the end of the proof in the example a small square occurred. In this way the end of proof is usually marked.

When proving an implication in the indirect way, we assume that the conclusion is false.

Reasoning therefrom brings us to contradiction with the theorem's assumption. The implication is proved.

indirect proof:

conclusion is false

↓

assumption is false

EXAMPLE 2 Prove the statement:

If number a is irrational then number $a + 3$ also is irrational.

Assumption: Number a is irrational.
Conclusion: Number $a + 3$ is irrational.

Proof

Suppose that number $a + 3$ is rational.

We assume the conclusion is not true.

So:

$a + 3 = \frac{p}{q}$ for some integers p and q.

Hence:

$a = \frac{p}{q} - 3, \quad \text{czyli} \quad a = \frac{p - 3q}{q}$

As p and q are integers, the number $p - 3q$ is an integer.

It follows therefrom that number a is a quotient of two integers, so it is a rational number. This contradicts the assumption.

From negation of the conclusion negation of the assumption was inferred. The statement is proved. □

PROBLEM Prove the statement:

If number a is irrational, number $\frac{a}{2}$ also is irrational..

Another way of proving indirectly is that we assume that the whole statement is not true and as a result of correct reasoning we come to the contradiction with some known mathematical fact. For example, such a well-known fact is the assertion: *Every composite natural number a is divisible by some prime number smaller of it.*

In the indirect proof that follows we are contradicting this rather obvious claim.

We will prove the theorem:

There are infinitely many prime numbers.

Proof

Suppose the prime numbers are finitely many.

We assume that the theorem is not true.

It follows that among prime numbers, you can indicate the largest. Let's denote it with letter p.

Thus, the list of all prime numbers in order from the smallest to the largest is as follows:

$2, \quad 3, \quad 5, \quad 7, \quad \ldots, \quad p$

Let us take number:

$L = 2 \times 3 \times 5 \times 7 \times \ldots \times p + 1$

The number L is greater than p, so it is not a prime number. It must be a composite number.

At the same time, the remainder of division of L by 2 is 1, the remainder of division of L by 3 is also 1 and the same is obtained by dividing L by each of the other primes: $5, 7, \ldots, p$.

Therefore, our composite number L is not divisible by any prime number. This is contradictory to the claim that each composite natural number is divisible by a prime number smaller of it.

Denying the claim ”prime numbers are infinitely many”, we have received a contradiction. Thus, we proved its truthfulness. □

In mathematics, one can also find statements in which there is a phrase: *if and only if*. A sentence in this form is called equivalence. We can write it using the symbol $\Leftrightarrow$.

Here are examples of equivalence:

A triangle is isosceles if and only if its two angles have the same measure.

$a^2 = b^2 \iff a = b \text{ or } a = -b$

Equivalence is a sentence in the form

$$\boldsymbol{p \Leftrightarrow q}$$

It reads:

p if and only if q

This sentence means that there are both implications:

$$p \Rightarrow q$$

$$q \Rightarrow p$$

To prove a theorem formulated in the form of equivalence, we have to prove two implications.

For example, to prove the following equivalence for natural numbers:

The product $a \times b$ is an even number if and only if number a is even or number b is even.

we have to prove the implication:

If the product $a \times b$ is an even number, number a is even or number b is even.

and the reverse implication:

If number a is even or number b is even, then the product $a \times b$ is an even number.

EXERCISE Prove each of the above implications.

Tip. Prove the first of these implications by the indirect method — assume that the conclusion is not true (that is, that both numbers a and b are odd) and prove that the assumption ($a \times b$ is an even number) cannot be true. Prove the second implication by the direct method.

Curiosity

In mathematics, a sentence is only called a theorem whose truthfulness has been proved.

In general, theorems are created in such a way that an observed regularity, e.g. concerning numbers or geometric figures, is formulated and then it is proved.

If someone conjectured and formulated a certain regularity but did not prove it, then we say that he made a hypothesis.

One of the most famous hypotheses in the history of mathematics was the hypothesis of Fermat. On the margins of a mathematical work Pierre de Fermat wrote that he can prove the following property of natural numbers:

For $n \geqslant 3$ there is no triple of positive natural numbers x, y, z satisfying the equation $x^n + y^n = z^n$.

He also wrote that the justification does not fit in the margins of the book. To this day, we do not know if Fermat knew the correct proof of this statement.

Proving Fermat's hypothesis has been attempted for over 350 years. It was not until 1994 that English mathematician Andrew Wiles found a proof. Since then on it may already be called the Fermat theorem.

Of course, not every hypothesis turns out to be a theorem. For example, another hypothesis put forward by Fermat turned out to be false:

For each natural number n the number $2^{2^n} + 1$ is prime number.

PROBLEMS

1. Prove that:

a) the sum of three consecutive natural numbers is divisible by 3,

b) the sum of two consecutive odd numbers is divisible by 4,

c) the sum of four consecutive odd numbers is divisible by 8.

2. Prove the claim.

a) The square of an even number is divisible by 4.

b) The product of an even and an odd number is an even number.

c) If a natural number is divisible by 6, it is divisible by 3.

d) The difference of the squares of two odd numbers is an even number.

3. Prove the claim.

a) For each real number x the number x^2+1 is positive.

b) For any real number x, the number x^2-4x+4 is non-negative.

c) For any real number x, the number x^2-2x+2 is a positive number.

4. Let n be a natural number. Prove that:

a) number n^2+n is even,

b) the last digit of $5n^2+5n$ is 0,

c) number $3n^2+3n$ is divisible by 6,

d) number $\frac{n^2}{2}-\frac{3n}{2}$ is an integer.

5. Prove the claim.

a) If a and b are rational numbers, then $a+b$ is rational number.

b) If $2a+3$ is a rational number, then a is a rational number.

c) If $\frac{a-7}{5}$ is an irrational number, then a is an irrational number.

6. Prove the claim. Assume that the last digit of the number under consideration is not 0.

a) If we subtract from a two-digit number the two-digit number made by reversing the order of digits of this number, then we get a number divisible by 9.

b) If from a three-digit number we subtract the number created by exchanging the digit of units with the digit of hundreds, we get a number divisible by 11.

7. Prove that a natural number n is a divisor of a natural number m if and only if $n \times m$ is the divisor of the number m^2.

Curiosity

In 1742, the Prussian mathematician Christian Goldbach put the following hypothesis:

Every even number greater than 2 is the sum of two prime numbers.

This hypothesis has not been confirmed or disproved until today, although for some time it was possible to win for it a prize of USD 1 million. In order to confirm the Goldbach hypothesis, one must give its proof, and in order to refute it, it is enough to give a counter-example, i.e. an even number, which does not fulfill the condition described.

8. Read the Curiosity. Prove that the given sentence is false, giving a counter-example.

a) The difference of two negative numbers is a negative number.

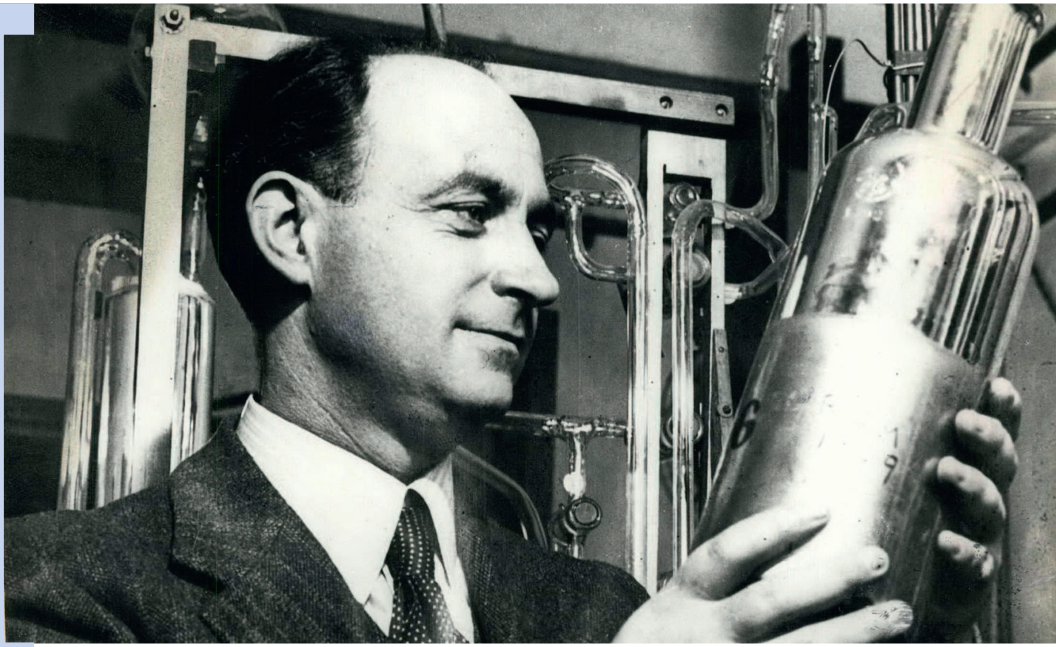
b) A number that is divisible by 4 and by 6 is divisible by 24.

c) The arithmetic mean of four even numbers is an even number.

d) If a quadrilateral has equal diagonals, it is a rectangle.

Powers and roots

Italian physicist Enrico Fermi was one of the greatest scholars of the 20th century. He liked to solve problems in which it was necessary to estimate unusual quantities, e.g.: "How many piano tuners are there in New York?", "How many bean grains will fit in the jar?", "How many drops of water do you need to fill the bathtub?". These types of questions are called Fermi's questions. When looking for answers to such questions, it is worth using exponential notation.



Powers with integer exponents ▪ Roots ▪ Powers with rational exponents ▪ Powers with real exponents

POWERS WITH INTEGER EXPONENTS

EXERCISE A Calculate: 6^1, $\left(1\frac{2}{3}\right)^3$, $1{,}3^0$, 0^5, 1^6, $0{,}7^2$.

Next to it, we remind you of the definition of the power with a natural exponent.

$$a^n$$

power's exponent — n

base of power — a

Note. The value of the power 0^0 is not defined, i.e., the notation 0^0 does not mean any number.

We assume that:

$$\boldsymbol{a^0 = 1} \qquad (\text{for } a \neq 0)$$

$$\boldsymbol{a^1 = a}$$

When n is natural number and $n > 1$, then:

$$\boldsymbol{a^n = \underbrace{a \times a \times a \times \ldots \times a}_{n \text{ factors}}}$$

By converting expressions that include powers, you can use the following identities. Each of these laws of operation is a theorem that can be proven.

LAWS OF OPERATIONS ON POWERS

If $a \neq 0$ and $b \neq 0$, then:

$$\boldsymbol{a^m \times a^n = a^{m+n}}$$

$$\boldsymbol{\frac{a^m}{a^n} = a^{m-n}}$$

$$\boldsymbol{(a^m)^n = a^{m \times n}}$$

$$\boldsymbol{(ab)^n = a^n b^n}$$

$$\boldsymbol{\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}}$$

Here's how you can prove that for a natural number n there is equality:

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n} \quad (a \neq 0 \text{ and } b \neq 0)$$

Proof

Let $n \in \mathbb{N}$, $a \neq 0$, $b \neq 0$.

If $n = 0$, then: $\left(\frac{a}{b}\right)^0 = 1 = \frac{1}{1} = \frac{a^0}{b^0}$

If $n = 1$, then: $\left(\frac{a}{b}\right)^1 = \frac{a}{b} = \frac{a^1}{b^1}$

If $n > 1$, then:

$$\left(\frac{a}{b}\right)^n = \underbrace{\frac{a}{b} \times \frac{a}{b} \times \ldots \times \frac{a}{b}}_{n \text{ factors}} = \frac{\overbrace{a \times a \times \ldots \times a}^{n \text{ factors}}}{\underbrace{b \times b \times \ldots \times b}_{n \text{ factors}}} = \frac{a^n}{b^n}$$ ◻

If we applied the formula $\frac{a^m}{a^n} = a^{m-n}$ for $m = 0$, we would get:

$$\frac{a^0}{a^n} = a^{0-n} = a^{-n}$$

It is known that for a natural number n and $a \neq 0$ there is equality:

$$\frac{a^0}{a^n} = \frac{1}{a^n}$$

Therefrom follows the equality: $a^{-n} = \frac{1}{a^n}$.

The obtained equality leads to the definition of powers with negative integer exponents just as in the frame.

Note that a^{-1} is the inverse of the number a, while a^{-n} is the inverse of the number a^n.

For a natural number n and $a \neq 0$ we assume that:

$$a^{-n} = \frac{1}{a^n}$$

For powers with negative integer exponents, the same laws of operations apply as for powers with natural exponents.

Here's how you can prove that for a natural number n there is equality:

$$\left(\frac{a}{b}\right)^{-n} = \frac{a^{-n}}{b^{-n}} \qquad (a \neq 0 \text{ and } b \neq 0)$$

Proof

Let $n \in \mathbb{N}$, $a \neq 0$, $b \neq 0$.

$$\left(\frac{a}{b}\right)^{-n} = \frac{1}{\left(\frac{a}{b}\right)^n} = \frac{1}{\frac{a^n}{b^n}} = \frac{1 \times \frac{1}{a^n}}{\frac{a^n}{b^n} \times \frac{1}{a^n}} = \frac{\frac{1}{a^n}}{\frac{1}{b^n}} = \frac{a^{-n}}{b^{-n}} \quad \square$$

EXERCISE B Prove the remaining laws of operation on powers with negative integer exponents.

EXERCISE C Write in the form of a power of number 7:

$7^{-1} \times 7^5 \qquad 7^{10} \div 7^{-4} \qquad (7^3)^{-2} \qquad (7^{-2})^{-2}$

Many seemingly very complicated calculations can be simplified by using the operations on powers.

EXAMPLE 1 Transform the expression using the rules of operations on powers.

a) $\frac{13^{-7} \times 13^3}{13^{-3}} = \frac{13^{-7+3}}{13^{-3}} = \frac{13^{-4}}{13^{-3}} = 13^{-4-(-3)} = 13^{-1} = \frac{1}{13}$

b) $\frac{49^{-5}}{7^{20}} = \frac{(7^2)^{-5}}{7^{20}} = \frac{7^{-10}}{7^{20}} = 7^{-10-20} = 7^{-30}$

c) $\frac{4^3 \times (2^{-3})^4}{2^{-10} \times 8} = \frac{(2^2)^3 \times (2^{-3})^4}{2^{-10} \times 2^3} = \frac{2^6 \times 2^{-12}}{2^{-7}} = \frac{2^{-6}}{2^{-7}} = 2^{-6-(-7)} = 2$

d) $\frac{3^{-15} - 3^{-17}}{2^{20}} = \frac{3^{-17}(3^2 - 1)}{2^{20}} = \frac{3^{-17} \times 8}{2^{20}} = \frac{3^{-17} \times 2^3}{2^{20}} = 3^{-17} \times 2^{-17} = 6^{-17}$

PROBLEM Convert to a power.

a) $\frac{(6^{-3})^5}{6^{-4}}$ b) $\frac{25^3}{5^9}$ c) $\frac{1}{9} \times 3^8 \times \frac{1}{3^{-4}}$ d) $\frac{2^{-16} + 2^{-14}}{5^{17}}$

$7\,000\,000$ kg $= 7 \times 10^6$ kg

$0{,}000004$ kg $= 4 \times 10^{-6}$ kg

EXERCISE D Convert to a power of 10.

$0{,}00001$ $\quad \frac{1}{1\,000\,000}$ $\quad 0{,}000001$ $\quad \frac{1}{0{,}001}$ $\quad 10\,000\,000$

When recording very large numbers or very small numbers, it is convenient to use exponential notation.

Exponential notation of numbers is the form:

$\boldsymbol{a \times 10^n}$, where $\boldsymbol{a} \in \langle \mathbf{1;10} \rangle$ and $\boldsymbol{n} \in \mathbb{Z}$

EXAMPLE 2 The average ice cube has a volume of $2{,}5\,\text{cm}^3$. How many such cubes could be made from an iceberg with a volume of $2 \times 10^6\,\text{m}^3$? Write the answer in exponential notation.

$$2{,}5\,\text{cm}^3 = 2{,}5 \times (10^{-2})^3\ \text{m}^3 = 2{,}5 \times 10^{-6}\,\text{m}^3$$

The volume of an ice cube is expressed in m^3.
$1\,\text{cm} = 10^{-2}\,\text{m}$, so $1\,\text{cm}^3 = (10^{-2})^3\ \text{m}^3$

$$\frac{2 \times 10^6\,\text{m}^3}{2{,}5 \times 10^{-6}\,\text{m}^3} = \frac{2}{2{,}5} \times 10^{6-(-6)} = 0{,}8 \times 10^{12} = 8 \times 10^{-1} \times 10^{12} = 8 \times 10^{11}$$

Ans. You can cut 8×10^{11} ice cubes from this iceberg.

PROBLEM A large spoon can get $15\,\text{cm}^3$ of water. Lake Śniardwy contains about $6{,}6 \times 10^8\,\text{m}^3$ of water. How many spoons of water contains this lake? Write the answer in exponential notation.

PROBLEMS

1. Calculate.

5^3 $\quad 5^{-3}$ $\quad (-5)^3$ $\quad (-5)^{-3}$ $\quad -5^3$ $\quad -5^{-3}$ $\quad 5^0$ $\quad -5^0$ $\quad (-5)^0$

2. Calculate.

a) 10^{-5} $\quad (-4)^{-2}$ $\quad (-2)^{-3}$ $\quad \left(\frac{2}{3}\right)^{-2}$ $\quad (-1{,}2)^{-1}$ $\quad (-0{,}1)^{-5}$ $\quad -0{,}02^{-4}$

b) $\left(\frac{3}{2}\right)^{-2}$ $\quad \left(1\frac{2}{3}\right)^{-1}$ $\quad (-1{,}5)^{-3}$ $\quad \left(-3\frac{1}{3}\right)^{-4}$ $\quad -0{,}01^{-3}$ $\quad -(-0{,}1)^5$ $\quad 1{,}1^{-2}$

3. Assume that the number a is positive. Which of the following numbers are also positive?

a^3 $\qquad -a^3$ $\qquad (-a)^3$ $\qquad a^{23}$ $\qquad \dfrac{a^{-2}}{a^{-5}}$ $\qquad (-a)^2$ $\qquad (-a)^{-2}$ $\qquad -(-a)^{31}$

4. Write the given numbers in the form of powers with negative exponents.

$\left(\frac{1}{3}\right)^4$ $\qquad \left(\frac{3}{8}\right)^3$ $\qquad \frac{1}{7^5}$ $\qquad 6^5$ $\qquad \left(2\frac{1}{4}\right)^7$

5. Which of the calculations are incorrect?

a) $2^{-3} = 8$	c) $(-3)^{-2} = \frac{1}{9}$	e) $\left(2\frac{1}{3}\right)^{-1} = 3\frac{1}{2}$	g) $\left(1\frac{3}{4}\right)^{-1} = 1\frac{4}{3}$
b) $\left(\frac{1}{2}\right)^{-2} = 2$	d) $(-1)^{-1} = 1$	f) $(-1)^{-4} = -1$	h) $0{,}2^{-1} = 5$

6. Write in the form of power.

a) $6^{-5} \div 6^{-2}$

b) $(8^{-3})^{-2}$

c) $15^{-3} \times 15^{-7}$

d) $3^{15} \times 3^3 \times 3^{-2}$

e) $4^{-7} \times 4^6 \times 4^{-2}$

f) $\dfrac{5^{-4} \times 5^3}{5^{-2}}$

g) $(7^{12} \div 7^3)^{-3} \times 7$

h) $(4^5)^3 \times 4^{-2} \div 4$

i) $\left[5^{-1} \div (5^2)^3\right] \times 5^3$

j) $(13^2 \div 13^5)^{-1} \times 13^{-2}$

k) $(4^2)^{-3} \div (4^{-4})^2 \div 4^{-1}$

l) $(2^3 \div 2^{-5})(2^{-4} \div 2^{-2})^{-1}$

7. Imagine that one person is slicing an A4 sheet in half and handing one half to the other person. The second person breaks the part of the card received in half and passes one of the parts to the third person, etc. What part of the card will the fifth person receive? Estimate (in square meters) the size of the piece that the eleventh person would receive.

8. What number the little square should be replaced with?

a) $10\,\text{m} = 10^{\square}\,\text{km}$

b) $100\,\text{kg} = 10^{\square}\,\text{t}$

c) $10\,\text{ml} = 10^{\square}\,\text{l}$

d) $10\,\text{g} = 10^{\square}\,\text{kg}$

e) $0{,}1\,\text{dag} = 10^{\square}\,\text{kg}$

f) $0{,}01\,\text{dm} = 10^{\square}\,\text{m}$

Prefix	Symbol	Fraction
deci-	d	10^{-1}
centi-	c	10^{-2}
milli-	m	10^{-3}
micro-	μ	10^{-6}
nano-	n	10^{-9}
pico-	p	10^{-12}

9. The table shows prefixes denoting parts of the basic units. Calculate:

a) how many nanograms is in a milligram,

b) how many picometers is in a decimeter,

c) how many micrometers is in a centimeter,

d) how many milliliters is in a centiliter,

e) how many nanoliters is in a centiliter.

10. Write the answers to the questions in exponential notation.

a) The Sun's volume is about $1{,}41 \times 10^{18}\ \mathrm{km}^3$. How many cubic meters is it?

b) The surface of the Sun is about $6{,}07 \times 10^{18}\ \mathrm{m}^2$. How many square kilometers is it?

c) The surface of a water drop is about $1{,}6 \times 10^{-10}\ \mathrm{mm}^2$. How many square meters is it?

Speed of light in the vacuum is about $300\,000\ \frac{\mathrm{km}}{\mathrm{s}}$.

11. a) The light-year is the distance that light travels in the vacuum during the year. Calculate and write in exponential notation how many kilometers has the light-year.

b) The diameter of our Galaxy is about 100 000 light-years. How many kilometers is it?

ROOTS

EXERCISE A Calculate.

a) $\sqrt{25}$ $\sqrt{\frac{4}{49}}$ $\sqrt{1{,}21}$ $\sqrt{1}$ $\sqrt{0}$

b) $\sqrt[3]{27}$ $\sqrt[3]{-\frac{8}{125}}$ $\sqrt[3]{0{,}064}$ $\sqrt[3]{-1}$ $\sqrt[3]{0}$

You already know that the square root is defined for non-negative numbers and is a non-negative number, and the cube root is defined for any number and can be a negative number. Similarly, we distinguish even and odd degree when defining roots.

Note. Root is also called radical.

root's degree

$$\sqrt[n]{a}$$

radicand

If k is an even natural number greater than 1, then for $a \geqslant 0$ we assume that:

$$\sqrt[k]{a} = b \iff (b^k = a \text{ and } b \geqslant 0)$$

If m is an odd natural number greater than 1, then for any number a we assume that:

$$\sqrt[m]{a} = b \iff b^m = a$$

EXERCISE B Calculate.

a) $\sqrt[4]{16}$ $\sqrt[5]{\frac{1}{32}}$ $\sqrt[5]{-100\,000}$ $\sqrt[7]{-1}$ b) $\sqrt{8^2}$ $\sqrt{(-8)^2}$ $\sqrt{39^2}$ $\sqrt{(-39)^2}$

Note that $\sqrt{(-5)^2}$ is 5, not −5. It is worth remembering that if we want to simplify the expression $\sqrt{a^2}$, and we do not know what is the sign of the number a, we must indicate that the result is non-negative, e.g. by using the absolute value sign:

$$\sqrt{a^2} = |a|$$

It is necessary to similarly proceed in the case of roots of higher even degrees, e.g. $\sqrt[4]{a^4} = |a|$, $\sqrt[6]{a^6} = |a|$.

The numbers $\sqrt{0}$, $\sqrt{1}$, $\sqrt{4}$, $\sqrt{9}$, $\sqrt{16}$, $\sqrt{25}$, $\sqrt{36}$, $\sqrt{49}$, $\sqrt{64}$ etc. are natural numbers. It can be proved that the square root of any natural number, which is not a square of a natural number, is an irrational number.

Theorem: ***Number $\sqrt{2}$ is irrational.***

Proof

We will carry out the proof by the direct method.

Suppose that $\sqrt{2}$ is a rational number.

We assume that the theorem is not true.

Then $\sqrt{2} = \frac{p}{q}$ for some natural numbers p and q.

Hence: $2 = \frac{p^2}{q^2}$

$2q^2 = p^2$

Consider the number $2q^2$. In its prime factorization number 2 occurs an odd number of times.

If in the factorization of q the two occurs m times, then in the factorization of q^2 it occurs $2m$ times. Thus, in the factorization of number $2q^2$ the two will occur $2m + 1$ times.

Consider the number p^2. In its prime factorization number 2 occurs an even number of times.

If in the factorization of p the two occurs k times, then in the factorization of p^2 it occurs $2k$ times.

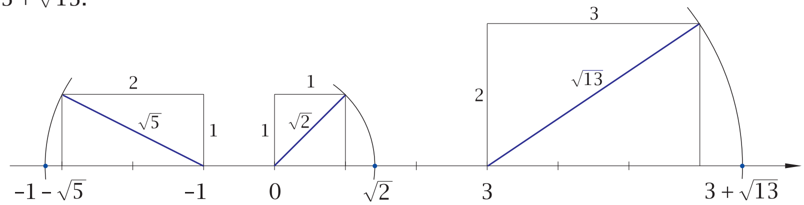
Thus, the numbers $2q^2$ and p^2 cannot be equal, because each number can be factorized into primes in an unambiguous way.

By denying the theorem, we have received a contradiction. Thus, we proved that the number $\sqrt{2}$ is not rational. □

In a similar way it can be proved that numbers: $\sqrt{3}$, $\sqrt{5}$, $\sqrt{6}$, $\sqrt{7}$, $\sqrt{8}$, $\sqrt{10}$ etc. are irrational numbers. It means that each of these numbers has the decimal infinite expansion non-periodic.

Curiosity

Irrational numbers appear in a natural way in geometry, for example the diagonal of the square with side of length 1 has the length $\sqrt{2}$, the diagonal of the rectangle with sides 1 and 2 has length $\sqrt{5}$ etc. Look how you can point exactly, on the number line, numbers $-1-\sqrt{5}$, $\sqrt{2}$ and $3+\sqrt{13}$.



Among the irrational numbers are also those that cannot be expressed with the help of roots of rational numbers. Such a number is, for example, π.

When transforming expressions with radicals we can use the following identities:

Rules of operation on radicals

For number k even:	For number m odd:
$\sqrt[k]{a^k} = \lvert a\rvert$	$\sqrt[m]{a^m} = a$
$\sqrt[k]{ab} = \sqrt[k]{a} \times \sqrt[k]{b}$ for $a \geqslant 0$ and $b \geqslant 0$	$\sqrt[m]{ab} = \sqrt[m]{a} \times \sqrt[m]{b}$
$\sqrt[k]{\frac{a}{b}} = \frac{\sqrt[k]{a}}{\sqrt[k]{b}}$ for $a \geqslant 0$ and $b > 0$	$\sqrt[m]{\frac{a}{b}} = \frac{\sqrt[m]{a}}{\sqrt[m]{b}}$ for $b \neq 0$
$\sqrt[k]{a^t} = (\sqrt[k]{a})^t$ for $a \geqslant 0$	$\sqrt[m]{a^t} = (\sqrt[m]{a})^t$
	$\sqrt[m]{-a} = -\sqrt[m]{a}$

Below we show how it can be proved that if $a \geqslant 0$, $b \geqslant 0$ and k is an even number, then:

$$\sqrt[k]{ab} = \sqrt[k]{a} \times \sqrt[k]{b}$$

Proof

Assume that $a \geqslant 0$, $b \geqslant 0$ and k is even.

Let's assume: $\sqrt[k]{a} = x$ and $\sqrt[k]{b} = y$, where $x \geqslant 0$ and $y \geqslant 0$.

Therefrom: $a = x^k$ and $b = y^k$

$$\sqrt[k]{ab} = \sqrt[k]{x^k \times y^k} = \sqrt[k]{(xy)^k} = \lvert xy\rvert = xy = \sqrt[k]{a} \times \sqrt[k]{b} \quad \square$$

Because $x \geqslant 0$ and $y \geqslant 0$, so $\lvert xy\rvert = xy$.

The other laws of operation on radicals can be proved in the same way.

EXERCISE C Prove one of the other laws of operation on radicals.

The laws of operation on radicals let you simplify certain expressions.

EXAMPLE 1 Calculate.

a) $(2\sqrt[3]{6})^3 = 2^3 \times (\sqrt[3]{6})^3 = 8 \times 6 = 48$

b) $\sqrt{2^{10}} = \sqrt{(2^5)^2} = 2^5 = 32$

c) $\sqrt[3]{2} \times \sqrt[3]{-4} = \sqrt[3]{-8} = -2$

d) $\sqrt{1\frac{3}{4}} \div \sqrt{7} = \sqrt{\frac{7}{4} \div 7} = \sqrt{\frac{1}{4}} = \frac{1}{2}$

PROBLEM Calculate.

a) $\left(2\sqrt[4]{3}\right)^4$ b) $(\sqrt[5]{-6})^{10}$ c) $\frac{\sqrt[3]{-80}}{\sqrt[3]{10}}$ d) $\sqrt{7^3} \times \sqrt{7}$

Sometimes it is convenient to write radicals in a different form — take out a factor of the radicand to the front of the radical symbol or do the opposite.

EXAMPLE 2 Take out a factor to the front of the radical sign.

a) $\sqrt{180} = \sqrt{9 \times 20} = \sqrt{9 \times 4 \times 5} = \sqrt{9} \times \sqrt{4} \times \sqrt{5} = 6\sqrt{5}$

b) $\sqrt[3]{-54} = -\sqrt[3]{27 \times 2} = -\sqrt[3]{27} \times \sqrt[3]{2} = -3\sqrt[3]{2}$

c) $\sqrt[4]{5^{13}} = \sqrt[4]{5^{12} \times 5} = \sqrt[4]{(5^3)^4} \times \sqrt[4]{5} = 125\sqrt[4]{5}$

d) $\left(\sqrt[5]{2}\right)^8 = \left(\sqrt[5]{2}\right)^5 \times \left(\sqrt[5]{2}\right)^3 = 2\sqrt[5]{8}$

PROBLEM Take out a factor to the front of the radical sign.

a) $\sqrt{450}$ b) $-\sqrt[3]{160}$ c) $\sqrt[5]{7^{16}}$ d) $\left(\sqrt[4]{3}\right)^7$

EXAMPLE 3 Bring the factor preceding the radical sign into the radicand.

a) $3\sqrt{5} = \sqrt{3^2} \times \sqrt{5} = \sqrt{9 \times 5} = \sqrt{45}$

b) $2\sqrt[3]{100} = \sqrt[3]{2^3} \times \sqrt[3]{100} = \sqrt[3]{8 \times 100} = \sqrt[3]{800}$

PROBLEM Bring the coefficient into the radicand.

a) $5\sqrt{2}$ b) $2\sqrt{6}$ c) $3\sqrt[3]{2}$ d) $5\sqrt[3]{4}$

When the root of a rational number appears in the denominator of an expression, we can transform it so that the denominator is a rational number. We say then that we remove irrationality from the denominator.

EXAMPLE 4 Remove irrationality from the denominator.

a) $\frac{4+\sqrt{2}}{\sqrt{2}} = \frac{(4+\sqrt{2})\times\sqrt{2}}{\sqrt{2}\times\sqrt{2}} = \frac{4\sqrt{2}+2}{2} = 2\sqrt{2}+1$

b) $\frac{5}{2\sqrt[3]{3}} = \frac{5\times\sqrt[3]{3}\times\sqrt[3]{3}}{2\sqrt[3]{3}\times\sqrt[3]{3}\times\sqrt[3]{3}} = \frac{5\sqrt[3]{9}}{6}$

c) $\frac{2}{2\sqrt{3}-1} = \frac{2\times(2\sqrt{3}+1)}{(2\sqrt{3}-1)(2\sqrt{3}+1)} = \frac{4\sqrt{3}+2}{12-1} = \frac{4\sqrt{3}+2}{11}$

We make use of the formula: $(a-b)(a+b) = a^2 - b^2$

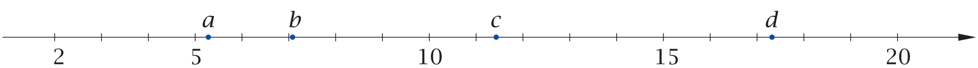
PROBLEM Remove irrationality from the denominator.

a) $\frac{6}{\sqrt{2}}$ b) $\frac{1-\sqrt{3}}{5\sqrt{3}}$ c) $\frac{8}{\sqrt[3]{6}}$ d) $\frac{1}{1+\sqrt{2}}$ e) $\frac{7}{\sqrt{3}-2}$

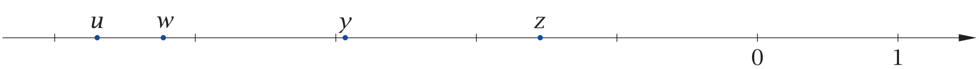
PROBLEMS

1. Some numbers are marked on the number line. Estimate these numbers and match each of them with the right letter.

a) $10\sqrt{3}$ $5\sqrt{2}$ $\sqrt{27}$ $\sqrt{130}$



b) $\sqrt[3]{-100}$ $2\sqrt[3]{-10}$ $-\frac{1}{2}\sqrt[3]{30}$ $-\sqrt[3]{25}$



2. Name two consecutive integers, one of which will be smaller than the given number, and the second — larger.

a) $\sqrt{109}$ **b)** $\sqrt{930}$ **c)** $\sqrt[3]{109}$ **d)** $\sqrt[3]{-930}$

3. Calculate.

a) $\sqrt{\sqrt{16}}$ **b)** $\sqrt{\sqrt[3]{0{,}000001}}$ **c)** $\sqrt[3]{\sqrt{(-8)^2}}$ **d)** $\sqrt{\sqrt{3\frac{13}{81}}}$

4. Calculate.

$\sqrt{(5^3)^2}$ $\sqrt{(0{,}1^2)^3}$ $\sqrt{11^4}$ $\sqrt{2^{10}}$ $\sqrt[3]{(7^2)^3}$ $\sqrt[3]{9^6}$ $\sqrt[3]{2^{12}}$

5. Write in the form of a power of 7.

$\sqrt{7^{16}}$ $\quad 7\sqrt[3]{7^{12}}$ $\quad (7\sqrt{7})^2$ $\quad 7\sqrt[3]{7}(\sqrt[3]{7})^2$ $\quad \left(\sqrt{7\sqrt[3]{7^6}}\right)^2$ $\quad \frac{1}{7^3} \times \sqrt{49} \times \sqrt{7^4}$

6. Take out a factor so that the radicand remained as small as possible.

a) $\sqrt{63}$ $\quad \sqrt{20}$ $\quad \sqrt{32}$ $\quad \sqrt{50}$

b) $\sqrt{98}$ $\quad \sqrt{108}$ $\quad \sqrt{99}$ $\quad \sqrt{160}$

c) $\sqrt[3]{32}$ $\quad \sqrt[3]{-40}$ $\quad \sqrt[3]{16}$ $\quad \sqrt[3]{-54}$

d) $\sqrt[3]{256}$ $\quad \sqrt[3]{-270}$ $\quad \sqrt[3]{-640}$ $\quad \sqrt[3]{1125}$

7. Which of the numbers is larger?

a) $7\sqrt{2}$ or $\sqrt{97}$

b) $5\sqrt{6}$ or $\sqrt{222}$

c) $10\sqrt[3]{7}$ or $\sqrt[3]{6789}$

d) $4\sqrt{5}$ or 9

e) $3\sqrt{11}$ or 10

f) $2\sqrt[3]{5}$ or 5

8. Remove irrationality from the denominator.

a) $\frac{8}{\sqrt{2}}$

b) $\frac{\sqrt{2}}{5\sqrt{3}}$

c) $\frac{5}{\sqrt[3]{7}}$

d) $\frac{1+\sqrt{2}}{3\sqrt{2}}$

e) $\frac{\sqrt[3]{5}+1}{2\sqrt[3]{5}}$

f) $\frac{6\sqrt{2}+2\sqrt{7}}{0{,}5\sqrt{7}}$

g) $\frac{\sqrt[3]{3}-\sqrt[3]{2}}{\sqrt[3]{6}}$

h) $\frac{\sqrt[3]{2}+\sqrt[3]{3}}{\sqrt[3]{4}}$

9. Prove that true is the sentence:

a) the inverse of number $\sqrt{2}$ is half that number,

b) the inverse of the number by 2 greater than $\sqrt{5}$ is the number by 2 less than $\sqrt{5}$.

POWERS WITH RATIONAL EXPONENTS

You can already calculate powers with integer exponents. You can also find the value of a power whose exponent is not an integer. Let's think, for example, what numbers may be hidden under the powers $5^{\frac{1}{3}}$, $5^{\frac{2}{3}}$. If the laws of operations on powers are to be kept, the following must be true.

$$\left(5^{\frac{1}{3}}\right)^3 = 5^{\frac{1}{3}\times 3} = 5^1 = 5 \qquad\qquad \left(5^{\frac{2}{3}}\right)^3 = 5^{\frac{2}{3}\times 3} = 5^2$$

As number 5 can be written in the form $\left(\sqrt[3]{5}\right)^3$, so there is equality:

$$5^{\frac{1}{3}} = \sqrt[3]{5}$$

Because the number 5^2 can be written in the form $\left(\sqrt[3]{5^2}\right)^3$, then true should be the equality:

$$5^{\frac{2}{3}} = \sqrt[3]{5^2}$$

In a similar way — with the help of radicals — we define powers with rational exponents.

If n, k are natural numbers, $n > 1$ and $k > 0$, we assume that:

$$a^{\frac{1}{n}} = \sqrt[n]{a} \qquad (\text{for } a \geqslant 0)$$

$$a^{\frac{k}{n}} = \sqrt[n]{a^k} \qquad (\text{for } a \geqslant 0)$$

$$a^{-\frac{k}{n}} = \frac{1}{a^{\frac{k}{n}}} = \frac{1}{\sqrt[n]{a^k}} \qquad (\text{for } a > 0)$$

EXERCISE a) Write down using radicals.

$7^{\frac{1}{4}} \quad 6^{\frac{3}{5}} \quad 10^{-\frac{2}{3}}$

b) Write down using powers.

$\sqrt{15} \quad \sqrt[3]{5} \quad \sqrt[5]{3^4}$

Note that the odd-degree roots are defined for all real numbers, including negative ones. However, we have defined powers with a rational exponent only in the case, when the base is non-negative. Because, if we accepted, for example, that $(-8)^{\frac{1}{3}} = \sqrt[3]{-8}$, then using the laws of operations we would receive the following contradiction:

$$(-8)^{\frac{1}{3}} = \sqrt[3]{-8} = -2 \quad \text{and} \quad (-8)^{\frac{1}{3}} = (-8)^{\frac{2}{6}} = ((-8)^2)^{\frac{1}{6}} = \sqrt[6]{64} = 2$$

EXAMPLE 1 Calculate.

a) $9^{\frac{1}{2}} = \sqrt{9} = 3$

b) $8^{\frac{2}{3}} = \sqrt[3]{8^2} = \left(\sqrt[3]{8}\right)^2 = 4$

c) $32^{-\frac{3}{5}} = \dfrac{1}{\sqrt[5]{32^3}} = \dfrac{1}{\left(\sqrt[5]{32}\right)^3} = \dfrac{1}{8}$

d) $16^{1,5} = 16^{\frac{3}{2}} = \sqrt{16^3} = \left(\sqrt{16}\right)^3 = 4^3 = 64$

e) $2^{\frac{6}{5}} = \sqrt[5]{2^6} = \sqrt[5]{2^5 \times 2} = 2\sqrt[5]{2}$

f) $8^{-1\frac{1}{3}} = \dfrac{1}{8^{\frac{4}{3}}} = \dfrac{1}{\sqrt[3]{8^4}} = \dfrac{1}{\left(\sqrt[3]{8}\right)^4} = \dfrac{1}{16}$

PROBLEM Calculate.

a) $10\,000^{\frac{1}{4}}$ b) $27^{\frac{2}{3}}$ c) $\left(\frac{1}{25}\right)^{1,5}$ d) $4^{-2\frac{1}{2}}$

The power with rational exponent is defined in such a way that the same laws of operations are met as for powers with integer exponents.

Using the identities next to, you can transform expressions in which there are powers with rational exponents.

Laws of operations on powers with rational exponents

$a^x \times a^y = a^{x+y}$ for $a \geqslant 0$

$\dfrac{a^x}{a^y} = a^{x-y}$ for $a > 0$

$(a^x)^y = a^{x \times y}$ for $a \geqslant 0$

$a^x \times b^x = (ab)^x$ for $a \geqslant 0,\ b \geqslant 0$

$\dfrac{a^x}{b^x} = \left(\dfrac{a}{b}\right)^x$ for $a \geqslant 0,\ b > 0$

EXAMPLE 2 Transform the expression.

a) $3^{\frac{5}{3}} \times 9^{\frac{5}{3}} = 27^{\frac{5}{3}} = \left(27^{\frac{1}{3}}\right)^5 = \left(\sqrt[3]{27}\right)^5 = 3^5 = 243$

b) $\left(8^{-\frac{5}{6}}\right)^2 = 8^{-\frac{5}{3}} = \left(\sqrt[3]{8}\right)^{-5} = 2^{-5} = \frac{1}{32}$

c) $3^{\frac{3}{2}} \times 3^{-\frac{3}{4}} = 3^{\frac{3}{2}-\frac{3}{4}} = 3^{\frac{3}{4}} = \sqrt[4]{27}$

d) $\dfrac{0{,}5^{-\frac{1}{8}}}{\left(2^{-\frac{1}{8}}\right)^7} = \dfrac{(2^{-1})^{-\frac{1}{8}}}{2^{-\frac{7}{8}}} = \dfrac{2^{\frac{1}{8}}}{2^{-\frac{7}{8}}} = 2^{\frac{1}{8}-(-\frac{7}{8})} = 2$

PROBLEM Calculate.

a) $64^{\frac{3}{4}} \div 4^{\frac{3}{4}}$ b) $27^{\frac{1}{2}} \times 9^{\frac{1}{4}}$ c) $(10^{-12})^{-\frac{1}{3}}$ d) $\dfrac{4^{-\frac{1}{3}}}{\left(2^{-\frac{1}{3}}\right)^2}$

Using powers with rational exponents, you can also transform some expressions with radicals.

EXAMPLE 3 Transform the expression.

a) $\sqrt[3]{2} \times \sqrt{2} = 2^{\frac{1}{3}} \times 2^{\frac{1}{2}} = 2^{\frac{1}{3}+\frac{1}{2}} = 2^{\frac{5}{6}} = \sqrt[6]{32}$

b) $\sqrt[3]{\sqrt{10}} = (\sqrt{10})^{\frac{1}{3}} = \left(10^{\frac{1}{2}}\right)^{\frac{1}{3}} = 10^{\frac{1}{6}} = \sqrt[6]{10}$

c) $\sqrt{3} \times \dfrac{1}{\sqrt[5]{9}} = 3^{\frac{1}{2}} \times (3^2)^{-\frac{1}{5}} = 3^{\frac{1}{2}-\frac{2}{5}} = 3^{\frac{1}{10}} = \sqrt[10]{3}$

d) $\sqrt[5]{8 \times \sqrt[3]{2}} = \left(2^3 \times 2^{\frac{1}{3}}\right)^{\frac{1}{5}} = \left(2^{\frac{10}{3}}\right)^{\frac{1}{5}} = 2^{\frac{2}{3}} = \sqrt[3]{4}$

PROBLEM Transform the expression.

a) $\sqrt{\sqrt{6}}$ b) $\dfrac{\sqrt{5}}{\sqrt[4]{5}}$ c) $\sqrt[4]{25} \times \sqrt{5}$ d) $\sqrt[3]{7\sqrt{7}}$

PROBLEMS

1. Calculate.

a) $8^{\frac{1}{3}}$ $16^{\frac{1}{4}}$ $0{,}01^{0{,}5}$ $25^{\frac{3}{2}}$ $0{,}027^{\frac{2}{3}}$ $64^{\frac{4}{3}}$

b) $144^{-\frac{1}{2}}$ $0{,}0016^{-0{,}25}$ $2{,}25^{-0{,}5}$ $0{,}125^{-\frac{2}{3}}$ $9^{-1{,}5}$ $\left(\frac{1}{16}\right)^{-0{,}75}$

2. Write using powers.

a) $\sqrt{13}$ $\sqrt[3]{2}$ $\sqrt{2^7}$ $\sqrt[5]{3^3}$ $\sqrt[4]{7^5}$ **b)** $\frac{1}{\sqrt{3}}$ $\frac{1}{\sqrt[3]{8^7}}$ $\left(\frac{1}{\sqrt{5}}\right)^2$ $\frac{1}{\sqrt[4]{2^3}}$ $\frac{1}{\sqrt[5]{0{,}7^2}}$

3. Present each of the given numbers in the form $a\sqrt[n]{b}$ so that the radicand be a natural number as small as possible. You can use one of the methods described in the box.

$5^{\frac{8}{7}}$ $7^{\frac{5}{3}}$ $\left(\frac{5}{4}\right)^{2,5}$

$0{,}1^{-\frac{11}{5}}$ $0{,}9^{-3,5}$

$$3^{\frac{15}{7}} = 3^{2\frac{1}{7}} = 3^{2+\frac{1}{7}} = 3^2 \times 3^{\frac{1}{7}} = 9\sqrt[7]{3}$$

$$3^{\frac{15}{7}} = \sqrt[7]{3^{15}} = \sqrt[7]{3^{14} \times 3} = \sqrt[7]{(3^2)^7 \times 3} = 9\sqrt[7]{3}$$

4. Present as an one power.

a) $5^{\frac{1}{2}} \times 5^{\frac{1}{4}}$ **e)** $5^{-\frac{1}{4}} \times 5^{-\frac{1}{2}} \times 5$ **i)** $2^{\frac{2}{3}} \times 8^{\frac{1}{6}}$

b) $3^{\frac{2}{3}} \div 3^{\frac{1}{6}}$ **f)** $3^{-\frac{4}{3}} \div 3^{\frac{1}{6}} \times 3^2$ **j)** $3^{-\frac{1}{4}} \times 9^{\frac{3}{4}}$

c) $7^{\frac{1}{5}} \times 7^{-\frac{1}{10}}$ **g)** $\left(7^{\frac{1}{4}}\right)^2 \times 7^{\frac{3}{4}}$ **k)** $25^{\frac{1}{3}} \div 5^{-\frac{2}{3}}$

d) $\left(6^{-\frac{2}{3}}\right)^{\frac{1}{5}}$ **h)** $2^{-\frac{1}{5}} \div \left(2^{\frac{2}{5}}\right)^{-\frac{1}{4}}$ **l)** $4^{-\frac{2}{5}} \div 2^{-\frac{3}{10}}$

5. a) Write each of the given numbers as a power of number 3.

$\sqrt[5]{9}$ $\frac{1}{\sqrt[3]{9}}$ $27\sqrt[3]{3}$ $9\sqrt[4]{27}$ $\frac{9}{\sqrt[5]{3}}$

b) Write each of the given numbers as a power of number 2.

$\sqrt[5]{2} \times \sqrt{\frac{1}{2}}$ $\frac{\sqrt[3]{2}}{\sqrt[4]{4}}$ $\sqrt{8} \times \sqrt[3]{4}$ $\sqrt[3]{\frac{1}{4}} \times \sqrt[6]{2}$ $\frac{\sqrt[3]{2}}{\sqrt{2}}$

c) Write each of the given numbers as a power of number 5.

$\sqrt{\sqrt{5}}$ $\sqrt[3]{\frac{1}{5}}$ $\frac{1}{\sqrt{\sqrt[3]{25}}}$ $\left(\sqrt[4]{\sqrt[3]{5}}\right)^2$ $\frac{\sqrt[3]{5}}{\sqrt{\sqrt{5}}}$

6. Prove equality.

a) $\sqrt[5]{7^3} \times 7^{\frac{3}{5}} = 7\sqrt[5]{7}$

b) $\frac{\sqrt[6]{\sqrt[7]{7} \times \sqrt[7]{7}}}{\sqrt[3]{7}} = \frac{1}{\sqrt[6]{7}}$

c) $2^{2,5} \times 16^{-0,5} = \frac{2}{\sqrt{2}}$

d) $\frac{4 \times (\sqrt[8]{2})^{-1}}{\sqrt[6]{8} \times \sqrt{\sqrt{2}}} = 0{,}5^{-1,125}$

e) $8^{0,25} \div \sqrt[4]{2} = 4^{\frac{1}{4}}$

f) $\frac{2^{-3,3}}{2^{-1,7}} = 0{,}25\sqrt[5]{4}$

POWERS WITH REAL EXPONENTS

You already know how to calculate a power of a positive base and rational exponent. We can also consider powers with positive bases, whose exponent is any real number, also irrational.

Consider, for example, what number could mean the record $3^{\sqrt{2}}$.

It is known that $\sqrt{2}$ is an irrational number, therefore it has an infinite non-periodic expansion.

$$\sqrt{2} = 1{,}414213562373095\ldots$$

The number $3^{\sqrt{2}}$ can be approximated by replacing the exponent with more and more accurate approximations of the number $\sqrt{2}$. Using the scientific calculator, we will get:

$$\sqrt{2} \approx 1{,}4, \text{ so } 3^{\sqrt{2}} \approx 3^{1{,}4} = 4{,}655536\ldots$$

$$\sqrt{2} \approx 1{,}41, \text{ so } 3^{\sqrt{2}} \approx 3^{1{,}41} = 4{,}706965\ldots$$

$$\sqrt{2} \approx 1{,}414, \text{ so } 3^{\sqrt{2}} \approx 3^{1{,}414} = 4{,}727695\ldots$$

$$\sqrt{2} \approx 1{,}4142, \text{ so } 3^{\sqrt{2}} \approx 3^{1{,}4142} = 4{,}728733\ldots$$

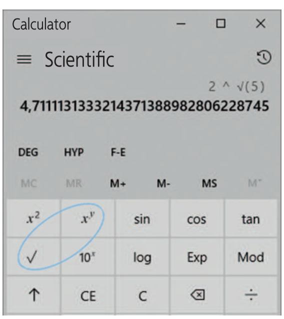
$$\sqrt{2} \approx 1{,}41421, \text{ so } 3^{\sqrt{2}} \approx 3^{1{,}41421} = 4{,}728785\ldots$$

The results obtained in this way would get closer and closer to the number $4{,}72880438783741494\ldots$ We assume that the power of $3^{\sqrt{2}}$ is equal this number, that is:

$$3^{\sqrt{2}} = 4{,}72880438783741494\ldots$$

Similarly, we can approximate other powers with irrational exponents.

In practice, the values of powers with irrational exponents can be estimated using the decimal approximations of exponents. E.g:



$$2^{\sqrt{5}} \approx 2^{2{,}236} \approx 4{,}71$$

$$2^{\pi} \approx 2^{3{,}14} \approx 8{,}82$$

$$\left(\frac{1}{2}\right)^{\sqrt{10}} \approx 0{,}5^{3{,}162} \approx 0{,}112$$

Some calculators allow you to calculate the powers of real exponents with even greater accuracy.

EXERCISE A Using the scientific calculator find an approximation of $2^{\sqrt{3}}$.

Even if we do not have a calculator, we can compare the values of powers.

EXERCISE B Determine which of the two given numbers is larger.

a) 5^3 or 5^4 b) $7{,}2^5$ or $7{,}2^4$ c) 2^{π} or 2^3 d) $6^{\sqrt{2}}$ or $6^{\sqrt{5}}$

EXERCISE C Determine which of the two given numbers is larger.

a) $\left(\frac{1}{5}\right)^3$ or $\left(\frac{1}{5}\right)^4$ b) $0{,}3^6$ or $0{,}3^5$ c) $\left(\frac{1}{2}\right)^{\pi}$ or $\left(\frac{1}{2}\right)^3$ d) $\left(\frac{2}{3}\right)^{\sqrt{2}}$ or $\left(\frac{2}{3}\right)^{\sqrt{5}}$

When we estimate a power, it should be remembered that the rule ”the larger the exponent the greater the power” applies only to powers with a base greater than 1.

If $a > 1$	If $0 < a < 1$
and	and
$x < y$	$x < y$
then	then
$a^x < a^y$	$a^x > a^y$

EXAMPLE 1 Using the inequality $3 < \sqrt{10} < 4$, estimate the given number.

a) $2^{\sqrt{10}}$

$\sqrt{10} > 3$, then:

$$2^{\sqrt{10}} > 2^3 = 8$$

$\sqrt{10} < 4$, then:

$$2^{\sqrt{10}} < 2^4 = 16$$

Hence:

$$8 < 2^{\sqrt{10}} < 16$$

b) $\left(\frac{1}{2}\right)^{\sqrt{10}}$

$\sqrt{10} > 3$, then:

$$\left(\frac{1}{2}\right)^{\sqrt{10}} < \left(\frac{1}{2}\right)^3 = \frac{1}{8}$$

$\sqrt{10} < 4$, then:

$$\left(\frac{1}{2}\right)^{\sqrt{10}} > \left(\frac{1}{2}\right)^4 = \frac{1}{16}$$

Hence:

$$\frac{1}{16} < \left(\frac{1}{2}\right)^{\sqrt{10}} < \frac{1}{8}$$

PROBLEM Estimate the number in the way shown above.

a) $3^{\sqrt{10}}$ b) $2^{\sqrt[3]{5}}$ c) $\left(\frac{1}{3}\right)^{\sqrt{7}}$

The laws of operations on powers with rational exponents are also valid for powers with real exponents.

EXAMPLE 2 Transform the expression.

a) $3^{-2\sqrt{2}} \times 3^{3\sqrt{2}} = 3^{-2\sqrt{2}+3\sqrt{2}} = 3^{\sqrt{2}}$

b) $\left(3^{\frac{3}{\sqrt{2}}}\right)^{\sqrt{2}} = 3^{\frac{3}{\sqrt{2}} \times \sqrt{2}} = 3^3 = 27$

c) $\dfrac{7^{2-\pi}}{7^{-\pi}} = 7^{(2-\pi)-(-\pi)} = 7^2 = 49$

PROBLEM Transform the expression.

a) $\dfrac{5^{\sqrt{7}-5}}{5^{\sqrt{7}}}$ b) $\left(8^{\frac{\sqrt{2}}{3}}\right)^{\sqrt{2}}$ c) $\left(3^{1+\sqrt{2}}\right)^{\sqrt{2}} \times 3^{-\sqrt{2}}$ d) $2^{4-2\pi} \times 4^{\pi}$

PROBLEMS

1. Arrange the given numbers in ascending order from the smallest to the largest.

a) $7^{-\sqrt{2}}$ 7^0 $7^{\frac{1}{2}}$ $7^{-2,1}$ 7^{π} 7^3 **b)** $\left(\frac{2}{3}\right)^{\sqrt{5}}$ $\left(\frac{2}{3}\right)^{\sqrt{3}}$ $\left(\frac{2}{3}\right)^{-1}$ $\left(\frac{2}{3}\right)^{0}$ $\left(\frac{2}{3}\right)^{\frac{1}{\pi}}$

2. Estimate which of the given numbers is larger: a or b.

a) $a = 2^{\sqrt{2}}$, $b = 2$ **c)** $a = 3$, $b = (\sqrt{3})^{\sqrt{3}}$

b) $a = 2^{\pi}$, $b = 8$ **d)** $a = \pi^{\sqrt{5}}$, $b = 9$

$\sqrt{2} \approx 1{,}41$
$\sqrt{3} \approx 1{,}73$
$\sqrt{5} \approx 2{,}24$
$\pi \approx 3{,}14$

3. Arrange the given numbers in descending order from the largest to the smallest.

$2^{5\sqrt{2}}$ $4^{\sqrt{3}}$ $\left(\frac{1}{2}\right)^4$ $\left(\sqrt[3]{2}\right)^{\pi}$ 32^0 $\left(\frac{1}{16}\right)^2$

4. Which of the given numbers is larger?

a) $2^{\sqrt{2}}$, $\left(\sqrt{2}\right)^3$ **b)** $27^{\frac{1}{\pi}}$, $\left(\frac{1}{9}\right)^{-\pi}$ **c)** $\left(\frac{1}{\pi}\right)^{\frac{1}{2}}$, $\left(\sqrt[3]{\pi}\right)^{-2}$

5. Calculate.

a) $\left(7^{\sqrt{2}}\right)^{\sqrt{2}}$ **c)** $\left(2{,}25^{-\frac{\sqrt{2}}{4}}\right)^{\sqrt{2}}$ **e)** $3^{3\sqrt{3}} \times 27^{1-\sqrt{3}}$

b) $(11^{\pi})^{-\frac{2}{\pi}}$ **d)** $\left(0{,}2^{\sqrt{6}}\right)^{\sqrt{\frac{3}{2}}}$ **f)** $\left(\frac{1}{5}\right)^{\sqrt{3}} \times 125^{\frac{1}{\sqrt{3}}}$

6. Find the number a that meets the given condition:

a) $\left(3^{\sqrt{2}}\right)^{a} = 3$ **b)** $3^{\sqrt{2}} \times 3^{a} = 3$ **c)** $\dfrac{3^{\sqrt{2}}}{3^{a}} = 3$

Logarithms

The magnitude of an earthquake is measured on the Richter scale. One of the strongest earthquakes registered in Poland had 4,8 degrees on this scale. It happened in 2010. In India, in 1950, an earthquake happened that was 8,7 Richter degrees. It does not mean that the earthquake in Poland was only about two times weaker of the one in India. Knowing the definition of the Richter scale and the properties of logarithms you can find out that the quake in India was 10 000 stronger than in Poland.

Definition of logartithm ▪ Logarithm features

DEFINITION OF LOGARITHM

Consider the following questions: *To what power should number 2 be raised to receive 32? To what power should number 2 be raised to receive 30?* These questions can be expressed in the form of equations:

$$2^x = 32 \qquad\qquad 2^x = 30$$

It is easy to say that the solution of the first equation is 5, because $2^5 = 32$. However, it is difficult to determine what number meets the equation $2^x = 30$. The number satisfying such an equation we will call the logarithm of the number 30 to base 2 and denote $\log_2 30$. You can check that $2^{4{,}91} \approx 30$, i.e. $\log_2 30 \approx 4{,}91$.

We define similarly logarithms to other bases.

For $a > 0$, $b > 0$ and $a \neq 1$
we assume that:

$$\mathbf{\log_a b = c \iff a^c = b}$$

The notation $\log_a b$ we read:
the logarithm of b to base a.

number which logarithm
in taken of

$$\log_a b$$

base
of logarithm

It can be said that number $\log_a b$ is the exponent of the power to which a should be raised to obtain number b.

Note that the number satisfying the equation $2^x = 32$ is the logarithm of number 32 to base 2. Because $2^5 = 32$, so $\log_2 32 = 5$.

EXAMPLE 1 Calculate.

a) $\log_3 81 = 4$, because $3^4 = 81$

b) $\log_{10} 0{,}001 = -3$, because $10^{-3} = 0{,}001$

c) $\log_8 2 = \frac{1}{3}$, because $8^{\frac{1}{3}} = \sqrt[3]{8} = 2$

d) $\log_{\frac{2}{3}} \frac{3}{2} = -1$, because $\left(\frac{2}{3}\right)^{-1} = \frac{3}{2}$

e) $\log_7 7 = 1$, because $7^1 = 7$

f) $\log_5 1 = 0$, because $5^0 = 1$

PROBLEM Calculate.

a) $\log_2 4,\ \log_{10} 10\,000$

b) $\log_7 \frac{1}{7},\ \log_{\frac{1}{3}} \frac{1}{27}$

c) $\log_6 6,\ \log_{\frac{2}{3}} 1,\ \log_{\frac{1}{4}} \frac{1}{4},\ \log_8 1$

d) $\log_5 5^4,\ \log_3 3^{12},\ \log_{\frac{1}{3}} \frac{1}{3^5}$

Logarithms are used in various fields of knowledge. Of particular importance is the logarithm to base 10, which we call the decimal logarithm. We write it in a simplified way. Instead of $\log_{10} b$, we write $\log b$.

$\log b$ means $\log_{10} b$

The decimal logarithms can be easily calculated when we have the suitable calculator. The key with the symbol *log* is used for this purpose.

EXERCISE Calculate using the calculator:

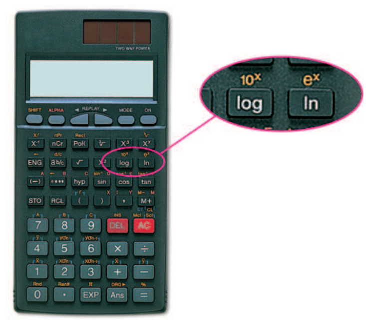
$\log 6 \quad \log 89 \quad \log 0{,}2 \quad \log 1500$

Curiosity

On the calculator — in addition to the *log* key used to calculate the decimal logarithms — you can also find the *ln* key. It is used to calculate logarithms, which are based to a certain irrational number called number e.

$$e = 2{,}71828182845904\ldots$$

The logarithm to base e is called natural logarithm; we write $\ln b$ instead of $\log_e b$.



Number e was introduced into mathematics in the eighteenth century by Swiss mathematician Leonard Euler. He noticed that the numbers: $\left(1+\frac{1}{2}\right)^2$, $\left(1+\frac{1}{3}\right)^3$, $\left(1+\frac{1}{4}\right)^4$, $\ldots$ approach to a certain number, which he called number e.

From the definition of logarithm it follows that for $a > 0$ and $a \neq 1$:

$$\log_a 1 = 0 \qquad \log_a a = 1 \qquad \log_a a^b = b$$

EXAMPLE 2 Calculate.

$$\log_5 \frac{\sqrt[3]{5}}{125} = \log_5 \frac{5^{\frac{1}{3}}}{5^3} = \log_5 5^{\frac{1}{3}-3} = \log_5 5^{-2\frac{2}{3}} = -2\frac{2}{3}$$

PROBLEM Calculate.

a) $\log_2 \frac{\sqrt{2}}{16}$ b) $\log_3 9\sqrt{3}$ c) $\log_5 \frac{25}{\sqrt{5}}$

When it is difficult to determine the value of a logarithm, you can use the method presented below.

EXAMPLE 3 Calculate $\log_9 27$.

$\log_9 27 = x$	The number searched for we denote x.
$9^x = 27$	$\log_a b = x \iff a^x = b$
$(3^2)^x = 3^3$	We transform the equation so that on both sides are powers with the same base.
$3^{2x} = 3^3$	We use the property that if powers of the same base are equal then exponents are equal.
$2x = 3$	
$x = \frac{3}{2}$	

Ans. $\log_9 27 = \frac{3}{2}$

PROBLEM Calculate.

a) $\log_8 \frac{1}{2}$ b) $\log_{\frac{1}{3}} 81$ c) $\log_{25} \sqrt{5}$ d) $\log_{0,1} \sqrt{10}$

Logarithms can be rational or irrational numbers. For example the numbers: $\log_2 4$, $\log_{\frac{1}{3}} 27$, $\log \sqrt{10}$ are rational, the numbers: $\log_2 5$, $\log_3 7$, $\log \frac{1}{3}$ are irrational. You can prove it.

Theorem: ***The number $\log_2 5$ is irrational.***

Proof
We will carry out the proof by the indirect method.

Let's assume that number $\log_2 5$ is rational.	We assume that the theorem isn't true.
So $\log_2 5 = \frac{p}{q}$ for some positive integers p and q.	Number $\log_2 5$ is positive. So, it can be assumed that p and q are also positive.

Than: $2^{\frac{p}{q}} = 5$

Thus: $\left(2^{\frac{p}{q}}\right)^q = 5^q$

Hence: $2^p = 5^q$

In the prime factorization of 2^p is only 2, and in the prime factorization of 5^q is only 5. Therefore, the numbers 2^p and 5^q cannot be equal.

By denying the claim, we have received a contradiction. Thus, we proved that the number $\log_2 5$ is irrational. □

From the history

The first to come up with the idea of introducing logarithm to calculation methods was Scotsman John Napier. He published his considerations in 1614, in the book ”Describing the Wonderful Principles of Logarithms.” Napier's logarithms differed a bit from those we use today, but they were such a huge advance in calculation that they immediately aroused enormous enthusiasm.

The term *logarithm* was introduced by Napier himself. It originated from the Greek words: *logos* — thinking and *arithmos* — counting.

The use of logarithms for astronomical calculations has helped Johannes Kepler to discover laws relating to planetary motion. Another well-known scholar Pierre Laplace claimed that ”the invention of logarithms (...) doubles the life of astronomers”.

PROBLEMS

1. **a)** Replace the given equality by one in the form $a^c = b$.

$\log_8 4 = \frac{2}{3}$ $\qquad$ $\log_7 1 = 0$ $\qquad$ $\log_5 \frac{1}{125} = -3$

b) Write numbers a, b, c and d as logarithms.

$5^a = 10$ $\qquad$ $\left(\frac{1}{4}\right)^b = 7$ $\qquad$ $0{,}02^c = \frac{2}{3}$ $\qquad$ $10^d = 0{,}6$

2. Find the numbers denoted with letters.

$\log_2 a = 4$	$\log_5 d = \frac{1}{2}$	$\log_2 g = -5$	$\log j = -0{,}5$
$\log_{\frac{1}{2}} b = 5$	$\log_{\sqrt{3}} e = \frac{1}{2}$	$\log_4 h = 2$	$\log k = 2$
$\log_{16} c = \frac{1}{4}$	$\log_{\frac{2}{5}} f = -1$	$\log i = \sqrt{3}$	$\log l = \frac{1}{3}$

3. Find the logarithms.

a) $\log_5 25$	**d)** $\log_7 \frac{1}{49}$	**g)** $\log_5 \sqrt{5}$	**j)** $\log_6 \sqrt[3]{36}$
b) $\log_3 27$	**e)** $\log_9 3$	**h)** $\log_7 \sqrt[4]{7}$	**k)** $\log_2 16$
c) $\log_6 \frac{1}{6}$	**f)** $\log_{1000} 10$	**i)** $\log_3 \sqrt[3]{3}$	**l)** $\log_8 2$

4. Calculate.

a) $\log 10^{\frac{2}{3}}$	**c)** $\log \frac{1}{10}$	**e)** $\log \sqrt{\frac{1}{10}}$	**g)** $\log 1$
b) $\log 100$	**d)** $\log 0{,}001$	**f)** $\log 10^8$	**h)** $\log \frac{\sqrt[3]{10}}{10}$

5. Calculate.

a) $\log_4 4$ $\log_7 1$ $\log_5 5^6$ $\log_3 3^{-7}$ $\log_6 6^{\frac{2}{3}}$

b) $\log_{\frac{2}{3}} \left(\frac{2}{3}\right)^{10}$ $\log_{\frac{1}{4}} \frac{1}{4}$ $\log_{\frac{3}{5}} \frac{5}{3}$ $\log_{\frac{3}{4}} 1$ $\log_{\frac{1}{7}} \left(\frac{1}{7}\right)^{-6}$

c) $\log_{\sqrt{5}} 1$ $\log_{\sqrt{3}} (\sqrt{3})^{-\frac{3}{4}}$ $\log_{\sqrt{2}} \left(\sqrt{2}\right)^{3}$ $\log_{\sqrt{6}} \sqrt{6}$ $\log_{\sqrt{7}} (\sqrt{7})^{5}$

6. Calculate.

a) $\log_5 5^{-\frac{2}{7}}$ $\log_5 (5^3)^7$ $\log_5 \left(5^{\frac{2}{3}}\right)^{-6}$ $\log_5 (5^{-4})^{\frac{1}{5}}$

b) $\log(10^{15})^{-4}$ $\log \frac{1}{100}$ $\log(\sqrt[3]{10})^2$ $\log \frac{1}{\sqrt[5]{100}}$

c) $\log_{\frac{1}{2}} 2^{-4}$ $\log_{\frac{1}{2}} (0{,}5^7)^8$ $\log_{\frac{1}{2}} 4^{13}$ $\log_{\frac{1}{2}} (\sqrt{2})^{18}$

7. Calculate.

a) $\log_3 9 \times \log_3 \frac{1}{3} \times \log_9 3 \times \log_3 \sqrt{3}$

b) $\log_2 \frac{1}{4} \times \log_{\frac{1}{2}} 2 \times \log_4 2 \times \log_2 \sqrt[3]{2}$

8. Calculate.

a) $\log_{0,1} 100$

b) $\log_2 \frac{1}{\sqrt{2}}$

c) $\log_4 8$

d) $\log_{\frac{1}{5}} \sqrt[4]{5}$

e) $\log_{\frac{2}{3}} \sqrt{1{,}5}$

f) $\log_4 \sqrt[3]{2}$

g) $\log_{\frac{1}{9}} 3\sqrt{3}$

h) $\log 10\sqrt{10}$

9. Calculate.

a) $\log_{\sqrt{5}} \sqrt[3]{5}$

b) $\log_{\sqrt{3}} 27$

c) $\log_{\sqrt{2}} 4\sqrt{2}$

d) $\log_{\sqrt[3]{6}} 6\sqrt{6}$

e) $\log_{\sqrt[3]{10}} \sqrt{10^5}$

f) $\log_{\sqrt{7}} \sqrt[3]{49}$

g) $\log_{\sqrt{11}} \frac{1}{\sqrt[5]{11^2}}$

h) $\log_{\sqrt[3]{5}} \frac{\sqrt[5]{5^3}}{5}$

10. Find the numbers labeled with letters.

a) $\log_a 125 = 3$

b) $\log_b 10\,000 = 2$

c) $\log_c \sqrt{2} = 1$

d) $\log_p 3 = \frac{1}{2}$

e) $\log_q 3^9 = 3$

f) $\log_r 7 = -\frac{1}{2}$

g) $\log_s 5 = -2$

h) $\log_t 0{,}0001 = -8$

i) $\log_w 0{,}5 = -\frac{1}{2}$

11. Prove that:

a) the number $\log_3 7$ is irrational,

b) the number $\log_2 6$ is irrational.

LOGARITHM FEATURES

The concept of logarithm is closely related to the concept of power. It is not surprising, therefore, that certain features of logarithms result from the laws of operations on powers.

EXERCISE A Check that the given equality is fulfilled.

a) $\log_2 4 + \log_2 8 = \log_2(4 \times 8)$ b) $\log 100 + \log 1000 = \log 100\,000$

Theorem on the logarithm of a product

If a, b and c are positive numbers and $a \neq 1$, then:

$$\log_a(bc) = \log_a b + \log_a c$$

Proof

We assume that $a > 0$, $b > 0$, $c > 0$ and $a \neq 1$.

Let: $\log_a b = x$ and $\log_a c = y$

So: $b = a^x$ and $c = a^y$

Therefore:

$$bc = a^x \times a^y$$
$$bc = a^{x+y}$$

We apply the features of powers.

Therefrom:

$$\log_a(bc) = x + y$$

We use the definition of logarithm.

Thus:

$$\log_a(bc) = \log_a b + \log_a c \quad \square$$

EXERCISE B Verify the truth of the given equality.

a) $\log_2 4 - \log_2 8 = \log_2 \frac{4}{8}$ b) $\log 100 - \log 1000 = \log 0{,}1$

Theorem on the logarithm of a quotient

If a, b and c are positive numbers and $a \neq 1$, then:

$$\log_a \frac{b}{c} = \log_a b - \log_a c$$

EXERCISE C Prove the theorem above. You can imitate the previous proof.

EXAMPLE 1 Calculate.

$$\log_5 10 + \log_5 7{,}5 - \log_5 3 = \log_5(10 \times 7{,}5) - \log_5 3 = \log_5 \frac{75}{3} = \log_5 25 = 2$$

PROBLEM Calculate.

a) $\log_6 9 + \log_6 4$ b) $\log_5 35 - \log_5 7$ c) $\log_2 12 + \log_2 6 - \log_2 9$

EXAMPLE 2 Assume that $\log 4 \approx 0{,}6$ and calculate an approximate value of $\log 250$.

$$\log 250 = \log \frac{1000}{4} = \log 1000 - \log 4 \approx 3 - 0{,}6 = 2{,}4$$

PROBLEM Calculate an approximate value of $\log 40$ and $\log 0{,}004$.

Please note that using the theorem on the logarithm of a product, we get:

$$\log_a b^2 = \log_a(b \times b) = \log_a b + \log_a b = 2\log_a b$$

EXERCISE D Prove that the equality $\log_2 c^3 = 3\log_2 c$ holds.

The equality in Exercise D concerned a logarithm of a power with a natural exponent. Similar equalities are true for logarithms of powers with any exponent.

Theorem on the logarithm of a power

If a and b are positive numbers and $a \neq 1$,
then for any real number p:

$$\boldsymbol{\log_a b^p = p \log_a b}$$

Proof

We assume that $a > 0$, $b > 0$ and $a \neq 1$.

Let: $\log_a b = x$ and $\log_a b^p = y$

So: $a^x = b$ and $a^y = b^p$

We apply the definition of logarithm.

Therefore:

$a^y = b^p = (a^x)^p = a^{xp}$

Therefrom: $a^y = a^{xp}$

$y = px$

Thus: $\log_a b^p = p \times \log_a b$ $\square$

EXAMPLE 3 Based on the fact that $\log_3 7 \approx 1{,}8$, determine an approximate value of the given number.

a) $\log_3 49 = \log_3 7^2 = 2\log_3 7 \approx 3{,}6$

b) $\log_3 9\sqrt[3]{7} = \log_3 9 + \log_3 7^{\frac{1}{3}} = 2 + \frac{1}{3}\log_3 7 \approx 2{,}6$

PROBLEM Based on the fact that $\log_2 10 \approx 3{,}32$, determine an approximate value of the number:

a) $\log_2 \sqrt{10}$ b) $\log_2 0{,}01$ c) $\log_2 2\sqrt[3]{100}$ d) $\log_2 \frac{\sqrt{10}}{4}$

EXAMPLE 4 Present as a logarithm.

a) $\frac{1}{2}\log_7 36 - 2\log_7 5 = \log_7 36^{\frac{1}{2}} - \log_7 5^2 = \log_7 6 - \log_7 25 = \log_7 \frac{6}{25}$

b) $2 - \log_5 2 = 2\log_5 5 - \log_5 2 = \log_5 5^2 - \log_5 2 = \log_5 (25 \div 2) = \log_5 12{,}5$

PROBLEM Present as a logarithm.

a) $2\log_6 3 + \frac{1}{3}\log_6 8$ b) $4 - 2\log_3 7$

PROBLEMS

1. Present the given expression as a logarithm.

a) $\log_2 3 + \log_2 x$

b) $\log_5 a - \log_5 4b$

c) $\log \frac{m}{9} + \log 3m$

d) $\log 7a - \log 4a - \log 3a$

e) $\log_7 \frac{n}{2} + \log_7 \frac{n}{2} - \log_7 n$

f) $\log_6 a + \log_6 5b - \log_6 8c$

g) $\log_3 2v + \log_3 v - \log_3 3v$

h) $\log x + \log \sqrt{x} - \log \sqrt{2x}$

2. Calculate.

a) $\log_2 80 + \log_2 0{,}1$

b) $\log_3 4{,}5 + \log_3 2$

c) $\log 2000 + \log \frac{1}{2}$

d) $\log_5 100 - \log_5 4$

e) $\log_2 7 - \log_2 56$

f) $\log_7 14 - \log_7 2\sqrt{7}$

g) $\log_2 6 + \log_2 12 + \log_2 \frac{4}{9}$

h) $\log 3 + \log 2 - \log 6$

i) $\log 4 - \log 5 + \log 125$

3. Present the given logarithm as the sum or difference of logarithms.

a) $\log_2(5x)$

b) $\log_{\frac{1}{2}} \frac{a}{3}$

c) $\log \frac{4x}{y}$

d) $\log_7 \frac{2}{st}$

e) $\log_{0,1} \frac{u}{7v}$

f) $\log (abc)$

g) $\log \frac{ab}{c}$

h) $\log \frac{1}{abc}$

4. Let a is such a number that $\log a = 2$. Calculate:

a) $\log a^{13}$ **c)** $\log \frac{1}{a^5}$ **e)** $\log \sqrt[5]{a^3}$ **g)** $\frac{(\log a)^2}{\log a^2}$

b) $\log a^{-7}$ **d)** $\log \sqrt[4]{a}$ **f)** $\log \frac{1}{\sqrt{a}}$ **h)** $\log a \times \log a^3$

5. Present the given expression in the form of a logarithm.

a) $3\log_2 a + \log_2 b$ **c)** $3\log_3(4n) - 2\log_3(5n)$ **e)** $\log 2k^5 - 5\log k^3 - 2\log k$

b) $2\log_5(4x) + 5\log_5 y$ **d)** $-2\log z - 4\log(2z)$ **f)** $\log \frac{1}{a} - 2\log a^2 + 3\log a$

6. The formula in the frame allows to convert a logarithm to the decimal logarithm. Assuming that $\log 2 \approx 0{,}301$ and $\log 25 \approx 1{,}398$, and applying that formula calculate:

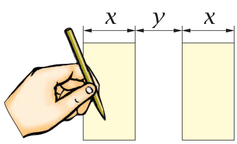
If $a > 0$, $b > 0$ and $a \neq 1$, then

$$\log_a b = \frac{\log b}{\log a}$$

a) $\log_2 25$ **b)** $\log_{25} 2$ **c)** $\log_{25} 4$ **d)** $\log_{\frac{1}{2}} 25$

Curiosity

Imagine an experiment in which the test subject has to alternately hit the pointer into two identical stripes located next to each other.



It's easy to understand that the time it takes to move the pointer and hit the strip depends on the width of the strips and the distance between them. In 1954, an American psychologist P.M. Fitts gave a formula that represents this relationship:

$$t = a + b\log\frac{2y}{x}$$

t — time (in seconds) needed move from one strip to the other

y — the distance (in cm) between the strips

x — the width (in cm) of each of the strips

a, b — constants characteristic for the given experiment

Fitts' formula was appreciated, among others, by the designers of computer system interfaces. Although it describes the simple truth that hitting the target is the easier it is the bigger and closer, it helps to decide whether it is better to place large icons on a larger surface, or small, but close to each other.

7. In an experiment similar to the one described in the above Curiosity, it was established that the constants a and b are: $a = 0{,}2$ s, $b = 0{,}5$ s. Thus:

$$t = 0{,}2 + 0{,}5\log\frac{2y}{x}$$

a) How much longer is time t in the case when $y = 3$ cm and $x = 7$ cm, from time t in the case when $y = 2$ cm and $x = 5$ cm?

● **b)** How much will time t be extended, when the distance between the strips will be doubled, and how much — when we increase it three times?

8. In 1938, American psychologist R.S. Woodworth carried out research on the speed of forgetting acquired knowledge when it is not perpetuated. It turned out that the phenomenon of forgetting can be described by the formula:

$$M = 100 - 35\log(t+1)$$

M — percentage of memorized facts

t — number of days that have elapsed since learning these facts

a) Calculate what part of facts was forgotten during the first 5 days and during 10 days of Woodworth's research. You can use the calculator.

● **b)** After what time was half of the facts forgotten?

Curiosity

In 1935, the seismologist Charles Richter came up with the idea of measuring the magnitude of earthquakes based on the amplitudes of oscillations caused by shocks. Initially, the measure of the amplitude (measured at a distance of 100 km from the epicenter) to the standard amplitude of 10^{-4} cm (equivalent to non-perceptible oscillations) was to be taken as the basis of this measure. It turned out, however, that he received numbers from very small to very large (from 0 to 800 000 000), which made it difficult to compare them. Therefore, he decided to assume the decimal logarithm of the numbers obtained as a measure of the magnitude of earthquakes. This is how the Richter scale was commonly adopted and used today.

The magnitude of an earthquake is calculated on this scale using the formula:

$$R = \log \frac{A}{A_0}$$

R — earthquake magnitude measured in degrees on the Richter scale

A — earthquake amplitude (in cm)

A_0 — reference amplitude (10^{-4} cm)

For example: if the magnitude of the earthquake was 100 times greater than the reference one $\left(\frac{A}{A_0} = 100\right)$, then $R = \log 100 = 2$. If the earthquake was 7 on the Richter scale, then it can be calculated from the above formula that it was 10 million times stronger than the quake with the reference amplitude A_0.

9. Read the curiosity.

a) How many degrees on the Richter scale had an earthquake with an amplitude of 1 cm?

b) The researchers estimate (based on the effects observed until today) that the strongest earthquake struck Poland on June 5, 1443 and had a strength of 6 on the Richter scale. Calculate the amplitude of this quake.

c) One of the strongest earthquakes in Poland with a magnitude 4,8 on the Richter scale recorded by seismographs took place in 2010. How many times was the amplitude of oscillations during this earthquake smaller than the amplitude during the earthquake that hit Japan in 2011 and had a strength of 9 on the Richter scale?